Lecture 04: Quotient space and the stabilizer group

Thursday, October 5, 2017 10:10 PM

Suppose $G \curvearrowright X$. Recall that we proved $G_x := \{g \in G \mid g \cdot x = x\}$ is a subgroup, and the following is a bijection from the set of left cosets of G_x to the orbit of x:

$$G/G_x \to G\cdot x, \quad gG_x \mapsto g\cdot x.$$

In particular, when one of these sets is finite we get

$$[G : G_x] = |G \cdot x|.$$

<u>Def.</u> We say $G \curvearrowright X$ freely if $G_x = \{1\}$ $\forall x \in X$.

<u>Example</u> $G \curvearrowright G$ by left translations is a free action.

So $\forall H \leq G$, $H \curvearrowright G$ freely. So $|H| = |Hg|$.

<u>Lagrange's theorem.</u> Suppose G is a finite group, and $H \leq G$. Then $|{}_H\backslash G| = |G|/|H|$.

<u>Pf.</u> We know that ${}_H\backslash G = \{Hg \mid g \in G\}$ is a partition of G. So

$$|G| = \sum_{Hg \in {}_H\backslash G} |Hg| \underset{\substack{\uparrow \\ \text{free action}}}{=} \sum_{Hg \in {}_H\backslash G} |H| = |H|\,|{}_H\backslash G|.$$

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Lecture 04: The Orbit-Stabilizer theorem

Tuesday, October 3, 2017 11:22 PM

As a corollary we get:

Theorem. Suppose G is a finite group and $G \curvearrowright X$.

ⓐ $\forall x \in X$, $|Gx| = [G:G_x] = |G|/|G_x|$.

ⓑ If X is finite, then $\frac{|X|}{|G|} = \sum_{Gx \in X/G} \frac{1}{|G_x|}$.

Pf. ⓐ We have already proved that there is a bijection from G/G_x to Gx. So by the Lagrange theorem, we get part ⓐ.

ⓑ Since X/G is a partition of X, we have

$$|X| = \sum_{Gx \in X/G} |Gx| = \sum_{Gx \in X/G} \frac{|G|}{|G_x|}. \qquad \blacksquare$$

Lemma. $G_{g\cdot x} = g\, G_x\, g^{-1}$; in particular, if G_x is finite, $\forall g \in G$, $|G_{g\cdot x}| = |G_x|$.

Pf. $g' \in G_{g\cdot x} \Leftrightarrow g'\cdot(g\cdot x) = g\cdot x$

$\Leftrightarrow (g^{-1} g' g)\cdot x = x \Leftrightarrow g^{-1} g' g \in G_x$

$\Leftrightarrow g' \in g\, G_x\, g^{-1}$. $\blacksquare$

. Let $X^g := \{x \in X \mid g\cdot x = x\}$ be the set of fixed points of $g \in G$.

Lemma. $g'\cdot X^g = X^{g' g g'^{-1}}$.

Lecture 04: Burnside's theorem on group actions

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Pf. $x \in X^g \iff g\cdot x = x \iff g\cdot g'^{-1}\cdot g'\cdot x = x$

$\iff (g' g g'^{-1})\cdot g'\cdot x = g'\cdot x \iff g'\cdot x \in X^{g' g g'^{-1}}$. ■

Corollary. If $|X| < \infty$, then $|X^g| = |X^{g' g g'^{-1}}|$; this means the number of fixed points of g is the same as the number of fixed points of any of g's conjugates.

Theorem. (Burnside) Suppose $|G|, |X| < \infty$ and $G \curvearrowright X$.

Then $|{}_G\backslash^X| = \frac{1}{|G|} \sum_{g\in G} |X^g|$. (The average of the number of fixed pts.)

Pf. Let $Y := \{(g,x) \mid g\cdot x = x\}$. Then

$$|Y| = \sum_{g\in G} |X^g| = \sum_{x\in X} |G_x| = \sum_{G\cdot x \in {}_G\backslash^X} \sum_{y\in G\cdot x} |G_y|$$

$$= \sum_{G x \in {}_G\backslash^X} |G_x|\,|G\cdot x| = \sum_{G\cdot x \in {}_G\backslash^X} |G|$$

$$= |G|\,|{}_G\backslash^X|.$$

$\Rightarrow |{}_G\backslash^X| =$ the average of the number of fixed points of elements of G. ■

Lecture 04: Transitive action

Wednesday, October 4, 2017 12:22 AM

<u>Ex.</u> Suppose $|G|, |X| < \infty$; and $G \curvearrowright X$ non-trivial transitively; that means $\left| {}_G\backslash X \right| = 1$ (there is only one G-orbit.). Then

$$\exists\, g \in G \setminus \{e\},\quad X^g = \varnothing$$

<u>Pf.</u> We know $\left| {}_G\backslash X \right| =$ average of $|X^g|$

Now, if $X^g \neq \varnothing$ for any g, then

$$1 = \text{the average of } |X^g| \geq \Big(|X| + \underbrace{1 + \cdots + 1}_{|G|-1}\Big) / |G|$$

$$= \frac{|X| + |G| - 1}{|G|}$$

$$\Rightarrow |G| \geq |X| + |G| - 1 \Rightarrow |X| \leq 1$$

$\Rightarrow G \curvearrowright X$ is trivial, which is a contradiction. ∎

<u>Ex.</u> Suppose $|G| < \infty$ and $H \lneq G$. Then

$$G \neq \bigcup_{g \in G} g H g^{-1}.$$

<u>Pf.</u> $G \curvearrowright G/H$ by the left translations;

It is a non-trivial transitive action;

$\Rightarrow \exists\, g_0 \in G$ s.t. $(G/H)^{g_0} = \varnothing \Rightarrow \forall g' \in G,\ g_0 g' H \neq g' H$

$\Rightarrow \forall g' \in G,\ g'^{-1} g_0 g' \notin H \Rightarrow \forall g' \in G,\ g_0 \notin g' H g'^{-1} \Rightarrow g_0 \notin \bigcup g H g^{-1}.$ ∎