Lecture 02: Parametrizing group actions

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At the end of the previous lecture we mentioned the following:

<u>Theorem A.</u> There is a bijection between

$$A_{G,X} := \{m : G \times X \to X \mid m \text{ is an action of } G \text{ on } X\}$$

and $\mathrm{Hom}(G, S_X)$, where $S_X = \{\sigma : X \to X \mid \sigma : \text{bijection}\}$.

In fact, the following is a bijection:

$$\Psi : A_{G,X} \to \mathrm{Hom}(G, S_X), \quad ((\Psi(m))(g))(x) := m(g,x).$$

And its inverse is given by

$$\Phi : \mathrm{Hom}(G, S_X) \to A_{G,X}, \quad (\Phi(f))(g,x) := (f(g))(x).$$

The statement of the theorem and its proof might look more complicated as they are! Essentially what we are doing is (as we mentioned in the previous lecture) <u>fixing g</u> and asking ourselves how it acts on X (for a given action m).

The way g acts on X is via the function $x \mapsto m(g,x)$

To write this officially we have write it this way

$$(\Psi(m))(g) \text{ maps } x \text{ to } m(g,x).$$

Lecture 02: Left translation and induced group actions

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We will see an outline of a proof of Theorem A. Now we see a few examples of group actions and a consequence of Theorem A.

<u>Example</u> (The left translation action) $G \curvearrowright G$ by the left translation; that means $g \cdot x := \underbrace{gx}_{\text{product in the group}}$.

($e \cdot x = ex = x$ and $$(g_1 \cdot (g_2 \cdot x)) = g_1 \cdot (g_2 x) = g_1(g_2 x) = (g_1 g_2) x = (g_1 g_2) \cdot x.$$)

<u>Example</u> (The left translation action) Suppose H is a subgroup of G. Then $G \curvearrowright G/H$ by the left translation; that means

$$g \cdot (g'H) := gg'H.$$

(Properties of group action can be easily checked.)

<u>Ex./Lem</u> (Induced group action) Suppose $G \curvearrowright X$ and $f : H \to G$ is a group hom. Then the following defines a left group action of H on X: $$h * x := f(h) \cdot x.$$

<u>Pf.</u>. $e * x = f(e) \cdot x = x$

the neutral element of G

$$\cdot\ h_1 * (h_2 * x) = f(h_1)\cdot(f(h_2)\cdot x) \underset{\substack{\uparrow\\ \boxed{\cdot \text{ is an action}}}}{=} (f(h_1)f(h_2))\cdot x \underset{\substack{\uparrow\\ \boxed{f \text{ is a gp hom.}}}}{=} f(h_1h_2)\cdot x = (h_1h_2)*x.$$

Lecture 02: Action by conjugation

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<u>Example</u> In the previous lecture we saw a general example:

$\mathrm{Symm}(X) \curvearrowright X$ where X is any object. For a group G, the group of symmetries is called <u>the automorphism group</u> of G, and it is denoted by $\mathrm{Aut}(G)$. So we get $\mathrm{Aut}(G) \curvearrowright G$ where $f \cdot g := f(g)$ for $f \in \mathrm{Aut}(G)$ and $g \in G$.

<u>Ex/Lem</u> (Acting by conjugation) The following defines a (left) action of G on G: $g * g' := g g' g^{-1}$. We say G acts on itself by conjugation.

<u>Pf.</u> One can check the properties of group action, but we present another argument:

<u>Recall</u>, for any $g \in G$, $c_g : G \to G$, $c_g(g') = g g' g^{-1}$ is an automorphism of G; c_g is called an <u>inner automorphism</u>

$c : G \to \mathrm{Aut}(G)$, $c(g) := c_g$ is a group homomorphism (why?)

So the action of $\mathrm{Aut}(G)$ on G induces an action of G on G

And here is the induced action $g * g' = c_g \cdot g' = c_g(g') = g g' g^{-1}$.

Lecture 02: Action on subgroups

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Ex./Lem. $\mathrm{Aut}(G) \curvearrowright \{H \mid H \text{ is a subgp of } G\}$

- For any n, $\mathrm{Aut}(G) \curvearrowright \{H \mid H \le G \text{ and } [G:H]=n\}$ (if $\neq \emptyset$)

Pf. • Let $f \in \mathrm{Aut}(G)$ and $H \le G$. Then $f(H)$ is a subgp of G;

- $I_G \cdot H = I_G(H) = H$;
- $f_1 \cdot (f_2 \cdot H) = f_1(f_2(H)) = (f_1 \circ f_2)(H) = (f_1 \circ f_2) \cdot H$.

• $[G:H]=n$ and $f \in \mathrm{Aut}(G)$. Then claim $[G:f(H)]=n$.

Suppose $G = \bigsqcup_{i=1}^{n} g_i H$ (disjoint union). It is enough to show ① $G = \bigcup_{i=1}^{n} f(g_i)\, f(H)$

② $f(g_i)\, f(H) \neq f(g_j)\, f(H)$

And both of these are consequ. of the fact that f is a bijection. ∎

Corollary. $G \curvearrowright \{H \mid H \le G\}$ and $\{H \mid H \le G \text{ and } [G:H]=n\}$, if $\neq \emptyset$, by conjugation; that means $g * H := g H g^{-1}$.

Pf. $c: G \to \mathrm{Aut}(G)$, $c(g) := c_g$ is a group hom. So the

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the actions of $Aut(G)$ on the sets of subgroups of G and subgroups of index n of G induce actions of G on those sets; and $g * H := c_g \cdot H = c_g(H) = gHg^{-1}$.

In view of the theorem mentioned at the beginning of the lecture, each one of these actions give us a group homomorphism from G to a symmetric group.

Lecture 02: Outline of the proof of Theorem A

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- We know that $S_X \curvearrowright X$. So any $f \in \mathrm{Hom}(G,X)$ induces a group action $G \curvearrowright X$:

$$g * x := f(g)\cdot x = \big(f(g)\big)(x)$$

This shows that $\Phi: \mathrm{Hom}(G, S_X) \to A_{G,X}$,

$$\Phi(f)(g,x) := \big(f(g)\big)(x)$$

is a well-defined function; that means $\Phi(f)$ is a group action.

- For $m \in A_{G,X}$ and $g \in G$, let $m_g : X \to X$, $m_g(x) = \underbrace{g \cdot x}_{m(g,x)}$.

<u>Step 1</u>. Show $m_{g_1} \circ m_{g_2} = m_{g_1 g_2}$.

<u>Step 2</u>. Show $m_e = I_X$.

Deduce $m_g \circ m_{g^{-1}} = m_{g^{-1}} \circ m_g = I_X$; and so $m_g \in S_X$.

<u>Step 3</u> Deduce $G \xrightarrow{\Psi} S_X$, $g \mapsto m_g$ is a well-defined group homomorphism.

<u>Step 4</u> Check that $\Psi \circ \Phi = I_{\mathrm{Hom}(G,S_X)}$ and $\Phi \circ \Psi = I_{A_{G,X}}$.