# Homework 5

Friday, November 3, 2017 12:00 AM

1. We say two finite groups $G_1$ and $G_2$ are <u>algebraically independent</u> if they do not have isomorphic simple quotients.

   (a) Prove that $G_1$ and $G_2$ are algebraically independent if and only if the following holds:

   $$(H \leq G_1 \times G_2 \text{ and } pr_1(H) = G_1 \text{ and } pr_2(H) = G_2) \Rightarrow H = G_1 \times G_2.$$

   (b) Suppose $G_i$ and $H$ are algebraically independent for $i = 1, 2$. Prove $G_1 \times G_2$ and $H$ are algebraically independent.

   (c) Suppose $\gcd(|G_1|, |G_2|) = 1$. Prove that $G_1$ and $G_2$ are algebraically independent.

   (d) Suppose $G_1$ and $G_2$ do not have isomorphic composition factors. Prove that they are algebraically independent.

2. (a) Suppose $G_1$ and $G_2$ are solvable groups and the following is a short exact sequence $1 \to G_1 \to G \to G_2 \to 1$. Prove that $G$ is solvable.

   (b) Suppose $A_1$ and $A_2$ are abelian groups and

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the following is a short exact sequence

$$1 \to A_1 \to G \to A_2 \to 1 .$$

Can we conclude that $G$ is nilpotent ?

3. Is $S_4$ solvable ? Is it nilpotent ?

4. Suppose $G$ is a group and $\{\gamma_i(G)\}_{i=1}^{\infty}$ is the lower central series of $G$. Recall that $[x,y] := x^{-1}y^{-1}xy$. We sometime write ${}^{x}y := x^{-1}yx$; and so $[x,y] = x^{-1}\,{}^{y}x$. A few useful formulas.

- $[x,y]^{-1} = [y,x]$.
- $[xy,z] = {}^{y}[x,z]\,[y,z]$
- $[[x,y],{}^{x^{-1}}z]\,[[z,x],{}^{z^{-1}}y]\,[[y,z],{}^{y^{-1}}x] = 1$. (Hall's equation)
- $[x^n,y] = {}^{x^{n-1}}[x,y] \cdot {}^{x^{n-2}}[x,y] \cdot \cdots \cdot {}^{x}[x,y] \cdot [x,y]$.

(a) Prove that $(xy)^n \equiv x^n y^n [y,x]^{\frac{n(n-1)}{2}} \pmod{\gamma_3(G)}$.

(Hint. ${}^{z}[y,x] \equiv [y,x] \pmod{\gamma_3(G)}$.

- Use induction and $y^n x = x y^n [y^n, x]$.)

(b) Suppose $N, M, L$ are normal subgroups of $G$. Prove

$$[[N,M],L] \le [[M,L],N]\,[[L,N],M].$$

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(c) Prove that, for any $m, n \in \mathbb{Z}^+$, we have

$$[\gamma_m(G), \gamma_n(G)] \subseteq \gamma_{m+n}(G).$$

(Hint. Use induction on $\min\{m,n\}$.)

(d) Let $f: \gamma_m(G)/\gamma_{m+1}(G) \times \gamma_n(G)/\gamma_{n+1}(G) \to \gamma_{m+n}(G)/\gamma_{m+n+1}(G)$,

$$f(x\gamma_{m+1}(G), y\gamma_{n+1}(G)) := [x,y]\gamma_{m+n+1}(G).$$

Prove that $f$ is a well-defined bilinear map, which means

$$f(\bar{x}_1 \cdot \bar{x}_2, \bar{y}) = f(\bar{x}_1, \bar{y}) f(\bar{x}_2, \bar{y}) \quad \text{and}$$

$$f(\bar{x}, \bar{y}_1 \cdot \bar{y}_2) = f(\bar{x}, \bar{y}_1) f(\bar{x}, \bar{y}_2).$$

(e) Let $L := \gamma_1(G)/\gamma_2(G) \oplus \gamma_2(G)/\gamma_3(G) \oplus \cdots$. So $L$ is an abelian groups. We use the plus sign $+$ to denote the group operation in $L$. Elements of $\gamma_i(G)/\gamma_{i+1}(G)$'s are called homogeneous elements of $L$. We let

$$[x\gamma_{n+1}(G), y\gamma_{m+1}(G)] := [x,y]\gamma_{m+n+1}(G),$$

and extend this bilinearly to a function $L \times L \to L$.

Use part (d) and convince yourself that this can be done.

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Prove that $[[\overline{x},\overline{y}],\overline{z}]+[[\overline{y},\overline{z}],\overline{x}]+[[\overline{z},\overline{x}],\overline{y}]=0$ in $L$.

(Remark. This is called the Jacobi identity; and this shows that $L$ is a Lie ring.)

(f) Show that $L$ is generated by $\gamma_1(G)/\gamma_2(G)$ as a Lie ring; this means you have to show

$$[L_1, L_n] = L_{n+1}$$

for any $n \in \mathbb{Z}^{\geq 1}$, where $L_n = \gamma_n(G)/\gamma_{n+1}(G)$.

Remark. Problem 4 presents an idea of translating some of the group theory problems to questions about Lie rings. This is the start of the profound proof of the Restricted Burnside Problem by E. Zelmanov. In the next problem, you can see an easy application of the above connection with Lie theory.

5. (a) Suppose $G = \langle g_1, \ldots, g_m \rangle$ is nilpotent and $o(g_i) < \infty$. Prove that $G$ is finite.

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(Hint. Show that $\gamma_1(G)/\gamma_2(G)$ is finite. Deduce that $\gamma_m(G)/\gamma_{m+1}(G)$ is finite for any $m\in\mathbb{Z}^{\geq 1}$.)

(b) Suppose $N$ is a nilpotent group. Prove that

$$T := \{g\in N \mid o(g)<\infty\}$$

is a subgroup.

(c) Let $D_\infty := (\mathbb{Z}/2\mathbb{Z})\ltimes_c \mathbb{Z}$ where $c: \mathbb{Z}/2\mathbb{Z}\to \operatorname{Aut}(\mathbb{Z})$

$(c(1+2\mathbb{Z}))(x) := -x$.

Prove that $D_\infty$ is solvable; but

$$T := \{x\in D_\infty \mid o(x)<\infty\}$$

is not a subgroup.

6. Let $A$ be a unital ring. $A$ is not necessarily commutative. Suppose $\mathfrak{a}$ is an ideal of $A$. Suppose $\mathfrak{a}^n=0$; that means $\forall\, x_1,\dots,x_n\in\mathfrak{a}$, $x_1\cdot x_2\cdot\dots\cdot x_n=0$. Let $G:=1+\mathfrak{a}$.

(a) Prove that $G$ is a subgroup of the group $U(A)$ of units of $A$. (Recall $U(A) := \{a\in A \mid \exists\, a'\in A,\ aa'=a'a=1\}$.)

(b) Prove that $\gamma_m(G)\subseteq 1+\mathfrak{a}^m$; and deduce that $G$ is nilpotent.