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. One of the important result in finite group theory is the following result of Burnside:

<u>Burnside's normal p-complement theorem.</u>

Suppose G is a finite group, $1 \neq P$ is a Sylow p-subgroup, and $P \subseteq Z(N_G(P))$. Then $\exists\ N \triangleleft G$ s.t. $|N| = |G/P|$.

This is an extremely useful theorem; for instance try to use this to give a short of a result we have proved earlier: a group G of order $p(p+1)$ has a normal subgroup of order p or $p+1$. (This is not part of the problem). In this problem you will see the powerful combination of this theorem with the Schur-Zassenhaus theorem:

1. Suppose $\gcd(n, \varphi(n)) = 1$, and G is a group of order n. Prove that a group of order n is cyclic.

(<u>Hint</u>: . <u>Arith. observations</u>: . $\gcd(n,\varphi(n))=1 \Rightarrow n$ is square-free

. $\left.\begin{array}{l}\gcd(n,\varphi(n))=1\\ m \mid n\end{array}\right\} \Rightarrow \gcd(m,\varphi(n)) = \gcd(m,\varphi(m)) = \gcd(n,\varphi(m)) = 1.$

. <u>Use strong induction on n</u>; and the mentioned theorems.)

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As we have seen in class $\mathrm{Aut}(G) \curvearrowright \{H \mid H \text{ is a subgp of } G\}$

$$f \cdot H := f(H).$$

Let $X := \{H \mid H \le G\}$. Then elements of $X^{\mathrm{Aut}(G)}$ are called <u>characteristic subgroups</u> of G; that means $H \le G$ is a characteristic subgroup if and only if, for any $f \in \mathrm{Aut}(G)$, $f(H) = H$. Convince yourself that any characteristic subgroup is a normal subgroup.

<u>2.</u> (a) Suppose $N \trianglelefteq G$ and K is a characteristic subgroup of N. Prove that $K \trianglelefteq G$.

(b) We say a group H is characteristically simple if its only char. subgroups are $\{1\}$ and H.

Suppose N is a <u>minimal</u> normal subgroup of G; that means, if $K \trianglelefteq G$ and $K \subsetneq N$, then $K = \{1\}$, and $N \ne \{1\}$.

Prove that N is characteristically simple.

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3. Suppose G is a group of order $2^k m$ where m is odd. Suppose a Sylow 2-subgroup P of G is cyclic. Prove that G has a characteristic subgroup of order m.

[Hint. Point 1. Use the case of $k=1$, and show that

$G \xrightarrow{\Phi} S_G \xrightarrow{\epsilon} \{\pm 1\}$ $\epsilon \circ \Phi$ is non-trivial.

Point 2. Suppose $\theta \in \mathrm{Aut}(G)$. Show that the cycle type of $\Phi(g)$ and $\Phi(\theta(g))$ are the same. Conclude that $\ker \epsilon \circ \Phi$ is a characteristic subgp of index 2.

Point 3. Use induction and deduce that there are char. subgps of order $2^i m$ for any $0 \le i \le k$.]

(Please do not use Burnside's normal p-compl. theorem)

4. ⓐ Suppose P is a Sylow p-subgp of G, and $P \lhd G$. Prove that P is a characteristic subgroup of G.

ⓑ Suppose $H \lhd G$ and $\gcd(|H|, [G:H]) = 1$. Prove that H is a characteristic subgroup of G.

5. In this problem, you prove that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ if $n \ge 7$. (All the automorphisms of S_n are inner.)

ⓐ Suppose $\varphi \in \mathrm{Aut}(S_n)$, $n \ge 5$, and φ sends transpositions to

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transpositions; that means $|\mathrm{supp}(\varphi(a\ b))| = 2$ for any $1 \le a < b \le n$.

Prove that φ is an inner automorphism.

[Hint ① Suppose τ_1 and τ_2 are two transpositions. Observe:

τ_1 and τ_2 do not commute if and only if $|\mathrm{supp}(\tau_1) \cap \mathrm{supp}(\tau_2)| = 1$.

② Any transposition gives us an edge in the complete graph with n vertices; by assumption φ induces a bijection on the edges of the complete graph. ① implies two edges with a common vertex are mapped to two edges with a common vertex. Use this to get a permutation σ on vertices.

③ Show that for any transposition τ, $\sigma\, \varphi(\tau)\, \sigma^{-1} = \tau$.]

Notice that in these parts φ is an arbitrary element of $\mathrm{Aut}(S_n)$

(b) Prove that $\varphi(\sigma_1)$ and $\varphi(\sigma_2)$ are conjugate if and only if σ_1 and σ_2 are conjugate.

(c) Let T_k be the set of permutations with cycle type $\underbrace{2,\ldots,2}_{k},\underbrace{1,\ldots,1}_{n-2k}$; for instance T_1 consists of transpositions. Show that

$$|T_k| = n(n-1)\cdots(n-2k+1)/_{k!\,2^k} \ge \frac{n(n-1)}{2} \quad \frac{(2k-2)!}{k!\cdot 2^{k-1}}.$$

(d) Prove that $\varphi(T_1) = T_k$ for some $1 \le k \le n/_2$. (Use part (b))

(e) Prove that $\varphi(T_1) = T_1$; and deduce that $\varphi \in \mathrm{Inn}(S_n)$.

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6. In this problem, you prove that $Aut(S_6) \neq Inn(S_6)$.

(In this problem you can use the fact that A_n is simple if $n \geq 5$)

(a) Show that S_5 has 6 Sylow 5-subgroups. Deduce that S_6 has a subgroup H which is isomorphic to S_5 and acts transitively on $\{1, 2, \cdots, 6\}$. And so $Fix(\sigma H \sigma^{-1}) = \emptyset$ for any $\sigma \in S_6$.

(b) Consider $S_6 \curvearrowright S_6/H$ by the left translations. Since $|H| = |S_5|$, we have $|S_6/H| = 6$. So the above action gives us a group homomorphism $\varphi: S_6 \to S_6$. Prove that φ is an isomorphism.

(c) Show that $Fix(\varphi(H)) \neq \emptyset$, and deduce φ is NOT an inner automorphism of S_6.