Homework 3

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1. Suppose p and q are distinct primes. Prove that a group of of order p^2q is not simple.

2. Prove that a group of order 36 is not simple.

(Hint. Suppose G is simple; find $|Syl_3(G)|$; consider $G \curvearrowright Syl_3(G)$ and show it should have a non-trivial kernel.)

3. Suppose $f_1, f_2 \in Hom(H, Aut(N))$. Suppose there is an isomorphism $\phi: H \ltimes_{f_1} N \to H \ltimes_{f_2} N$ such that $\circledast$ is a commuting diagram.

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, "="] & H \ltimes_{f_1} N \arrow[r] \arrow[d, "\phi"] & H \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & N \arrow[r] & H \ltimes_{f_2} N \arrow[r] & H \arrow[r] & 1 \end{tikzcd} \quad \circledast$$

Let $\sigma: H \to Aut(N)$ be the following function: $\sigma(h) = f_2(h) \circ f_1(h)^{-1}$.

ⓐ Prove that $\sigma(h) \in Inn(N)$ for any $h \in H$.

ⓑ Prove that, $\forall\, h_1, h_2 \in H$,

$$\sigma(h_1 h_2) = \sigma(h_1) \circ f_1(h_1) \circ \sigma(h_2) \circ f_1(h_1)^{-1}$$

(1-cocycle relation)

(Hint ⓐ Show $\phi(h,1) = (h, n(h))$; Consider $\phi((h,1)(1,n)(h,1)^{-1})$.

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4. ⓐ Show that the group $\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})$ of automorphism of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to the group $(\mathbb{Z}/n\mathbb{Z})^\times$ of units of the ring $\mathbb{Z}/n\mathbb{Z}$. (Recall that

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{ x \in \mathbb{Z}/n\mathbb{Z} \mid \exists y \in \mathbb{Z}/n\mathbb{Z},\ \underbrace{xy = 1}_{\text{in } \mathbb{Z}/n\mathbb{Z}} \}$$

And $a + n\mathbb{Z} \in (\mathbb{Z}/n\mathbb{Z})^\times \iff \gcd(a,n) = 1$.)

ⓑ Prove that a semidirect product $\mathbb{Z}/m\mathbb{Z} \ltimes \mathbb{Z}/n\mathbb{Z}$ is definitely abelian if and only if $\gcd(m, \varphi(n)) = 1$, where $\varphi(n)$ is the Euler φ-function. (Recall that $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$.)

5. Suppose $N_1, \dots, N_k$ are normal subgroups of G, and, for any i, $N_i \cap N_1 \cdots N_{i-1} N_{i+1} \cdots N_k = \{1\}$. Prove that

$$N_1 \times N_2 \times \cdots \times N_k \longrightarrow N_1 \cdot N_2 \cdot \cdots \cdot N_k$$
$$(x_1, x_2, \dots, x_k) \longmapsto x_1 \cdot x_2 \cdot \cdots \cdot x_k$$

is a group isomorphism.

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6. Suppose in a group G the following property holds:

$$(*) \quad H \lneq G \implies H \lneq N_G(H).$$

ⓐ Prove that all the Sylow subgroups are normal.

Deduce that $\forall p \mid |G|$, there is a unique Sylow p-subgp P_p.

ⓑ Prove that $G \simeq \prod_{p \mid |G|} P_p$.

(Hint: • If $P \in Syl_p(G)$, what do we know about $N_G(N_G(P))$?

• Use previous problem.)

7. Suppose A is an abelian normal subgroup of G. Let $H := G/A$.

Suppose $G = \bigsqcup_{i=1}^{m} g_i A$ (so $|H| = m$). Let $h_i := g_i A \in H$.

For any i, j, $g_i A \cdot g_j A = g_{k(i,j)} A$ for some $k(i,j)$

$\Rightarrow g_k^{-1} g_i g_j \in A$. So we get a function

$$c: H \times H \to A, \quad c(h_i, h_j) := g_{k(i,j)}^{-1} g_i g_j.$$

(c depends on the choice of representatives g_i's; think about g_i's as a section; that means $s: H \to G$, $s(h_i) = g_i$; and notice $s(h) A = h$.)

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Notice that $G \curvearrowright A$ by conjugation, and, since A is abelian, A is in the kernel of this action; this implies $H = G/A$ acts on A. For $h = gA$ and $a \in A$, we let ${}^{h}a := gag^{-1}$.

(so ${}^{h_1}({}^{h_2}a) = {}^{h_1h_2}a$ as it is an action; and

${}^{h}(a_1a_2) = {}^{h}a_1\,{}^{h}a_2$.)

(a) Prove that for any $h_1, h_2, h_3 \in H$ we have

$$c(h_1h_2, h_3)\; {}^{h_3^{-1}}c(h_1,h_2) = c(h_1, h_2h_3)\, c(h_2,h_3).$$

[It is better to use the section $s: H \to G$, $s(h_i) = g_i$; then

$s(h_1)\, s(h_2) = s(h_1h_2)\, c(h_1,h_2)$.

$(s(h_1)\, s(h_2))\, s(h_3) = s(h_1)\,(s(h_2)\, s(h_3))$

$\overset{?}{\Rightarrow}$ $s(h_1h_2)\, c(h_1,h_2)\, s(h_3) = s(h_1)\, s(h_2h_3)\, c(h_2,h_3)$

$\overset{?}{\Rightarrow}$ $s(h_1h_2)\, s(h_3)\; {}^{h_3^{-1}}c(h_1,h_2) = s(h_1h_2h_3)\, c(h_1,h_2h_3)\, c(h_2,h_3)$

$\overset{?}{\Rightarrow}$ $s(h_1h_2h_3)\, c(h_1h_2, h_3)\; {}^{h_3^{-1}}c(h_1,h_2) = s(h_1h_2h_3)\, c(h_1, h_2h_3)\, c(h_2,h_3)$.]

(This is called the 2-cocycle relation.)

(b) Prove that the short exact sequence $1 \to A \to G \to H \to 1$ splits if and only if $\exists$ a function $\alpha: H \to A$ such that

$$c(h_1,h_2) = {}^{h_2^{-1}}\alpha(h_1)\, \alpha(h_2)\, \alpha(h_1h_2)^{-1}$$

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[Hint. ($\Rightarrow$) Suppose $\psi: H \to G$ is the splitting homomorphism. Then

$\forall h \in H$, $\psi(h)A = s(h)A$. Let $\alpha: H \to A$, $\alpha(h) := \psi(h)^{-1} s(h)$.

Use $s(h_1)s(h_2) = s(h_1h_2)c(h_1,h_2)$ to check the relation.

($\Leftarrow$) Let $\psi(h) := s(h)\alpha(h)^{-1}$. Use the given relation to show $\psi: H \to G$

is a group hom. And notice $\psi(h)A = h$.]

(This is called the 2-boundary relation.)

(c) Suppose $\gcd(|A|, |H|) = 1$. Prove that a 2-cocyle $c: H \times H \to A$

is a 2-boundary; and so $1 \to A \to G \to H \to 1$

splits. (The abelian case of the Schur–Zassenhaus theorem.)

[Hint: The trick is "taking average"; in this part, proof would be more

clear if we use $+$ for the operation in A (notice that A is abelian.). So

c satisfies: $c(h_1h_2, h_3) + {}^{h_3^{-1}}c(h_1, h_2) = c(h_1, h_2h_3) + c(h_2, h_3)$. ⊛

Let $\alpha(h) := \frac{1}{|H|} \sum_{h_1 \in H} c(h_1, h)$ (why does it make sense? Here is where we are using $\gcd(|A|, |H|) = 1$.)

⊛ implies (why?)

$$\underbrace{\frac{1}{|H|}\sum_{h_1\in H} c(h_1h_2, h_3)}_{\alpha(h_3)} + \underbrace{{}^{h_3^{-1}}\left(\frac{1}{|H|}\sum_{h_1\in H} c(h_1, h_2)\right)}_{{}^{h_3^{-1}}\alpha(h_2)} = \underbrace{\frac{1}{|H|}\sum_{h_1\in H} c(h_1, h_2h_3)}_{\alpha(h_2h_3)} + c(h_2, h_3)$$

$$\alpha(h_3) + {}^{h_3^{-1}}\alpha(h_2) = \alpha(h_2h_3) + c(h_2, h_3).]$$