Homework 2

Friday, October 13, 2017   12:10 AM

1. (Double cosets) Suppose G is a group, and $H, K \leq G$. For any $g \in G$, let $HgK := \{hgk \mid h \in H, k \in K\}$. Notice that $H \curvearrowright G/K$ by left translations, and $H\backslash G \curvearrowleft K$ by right translations. Convince yourselves that

$$HgK = \text{union of elements of the } H\text{-orbit of } gK$$

and

$$HgK = \text{union of elements of the } K\text{-orbit of } Hg.$$

(a) Show that $\{HgK \mid g \in G\}$ is a partition of G.

This partition is denoted by $H\backslash G/K$.

(b) Show that there are bijections between the quotient spaces

$$H\backslash(G/K), \quad (H\backslash G)/K, \quad \text{and} \quad H\backslash G/K.$$

(c) Show that

$$H/H\cap gKg^{-1} \longrightarrow HgK/K,$$
$$h(H\cap gKg^{-1}) \longmapsto hgK$$

is a bijection; in particular, if $|G|<\infty$, then

$$|HgK| = |K||H| / |H\cap gKg^{-1}|.$$

(d) Let $G = SL_2(\mathbb{Z}/p\mathbb{Z})$ and $B = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \;\middle|\; a \in (\mathbb{Z}/p\mathbb{Z})^\times, b \in \mathbb{Z}/p\mathbb{Z} \right\}$. Find $|B\backslash G/B|$.

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(Hint For part ⓓ notice that $B \curvearrowright$ projective space $\mathbb{P}(\mathbb{F}^2)$ has two orbits: $\{[(1,0)]\}$ and $\{[(a,1)] \mid a \in \mathbb{F}\}$ ↳ any field

And $SL_2(\mathbb{F})/B$ is in bijection with the projective space $\mathbb{P}(\mathbb{F}^2)$.

Here $\mathbb{P}(\mathbb{F}^2) = \{[(a,b)] \mid (a,b) \in \mathbb{F}^2 \setminus \{(0,0)\}\}$ where $[(a,b)]$ is the line which passes through $(0,0)$ and (a,b).)

2. Suppose G is a group, and $|G| = p(p+1)$ where p is an odd prime. Suppose G has more than 1 Sylow p-subgroup. Prove that p is a Mersenne prime; that means $p = 2^n - 1$ for some positive integer n.

(Hint. Go over the proof presented in class, use Cauchy's theorem, and the fact that $2 \mid p+1$.)

3. Suppose G is a finite group, and $N \trianglelefteq G$. Let $P \in \mathrm{Syl}_p(N)$. Prove that $G = N_G(P)\, N$.

(Hint. Show that $G \curvearrowright \mathrm{Syl}_p(N)$ by conjugation; and then use Sylow's 2nd theorem.)

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4. Suppose $G \curvearrowright X$ transitively. Prove that the kernel of this group action is the normal core $\mathrm{cor}(G_x)$ of the stabilizer group of a point $x \in X$.

5. Suppose G is a group of order $pq\ell$ where $p, q,$ and ℓ are distinct prime numbers. Prove that G has a normal subgroup of prime order.

(Hint. Using the contrary assumption, show

$$|\mathrm{Syl}_\ell G| = pq, \quad |\mathrm{Syl}_q G| \ge p, \quad |\mathrm{Syl}_p G| \ge q.$$

And get a lower bound larger that $|G|$ for

$$|\{g \in G \mid o(g) \text{ is either } p, \text{ or } q, \text{ or } \ell\}|.)$$

6. Suppose G is a finite p-group and $\{e\} \ne N \trianglelefteq G$. Prove that $N \cap Z(G) \ne \{e\}$.

7. Suppose G is a finite group, $H \trianglelefteq G$, and p is a prime factor of $|H|$.

(a) Suppose $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(H)$. Prove that $\exists g \in G$ s.t. $Q = gPg^{-1} \cap H$.

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ⓑ Prove that the following is a well-defined surjective function

$$\begin{aligned} \mathrm{Syl}_p(G) &\xrightarrow{\Phi} \mathrm{Syl}_p(H), \\ P &\longmapsto P \cap H. \end{aligned}$$

ⓒ For $P \in \mathrm{Syl}_p(G)$, show that $G = N_G(P \cap H)\, H$, and conclude $|\mathrm{Syl}_p(G)| = \dfrac{|N_G(P\cap H)|}{|N_G(P)|} \cdot \dfrac{|H|}{|N_H(P\cap H)|}$.

ⓓ Prove that $|\mathrm{Syl}_p(H)| \mid |\mathrm{Syl}_p(G)|$.