Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	8	
5	10	
6	12	
Total:	60	

1. Let G be a finite simple group. Suppose $|\mathrm{Syl}_p(G)| = p+1$ for a prime p and $P \in \mathrm{Syl}_p$. Prove that
   (a) (3 points) G can be embedded into the symmetric group S_{p+1}.
   (b) (2 points) Prove $p^2 \nmid |G|$.
   (c) (2 points) $C_G(P) = P$.
   (d) (3 points) $|G| | p(p^2-1)$.

2. (10 points) Let P be a p-subgroup of a finite group G, and

$$f(P) := |\{Q \in \mathrm{Syl}_p(G) |\ P \subseteq Q\}|.$$

   Prove that $f(P) \equiv 1 \pmod p$.

3. (10 points) Let G be a finite group. Every proper subgroup of G of order divisible by p is a p-group. Prove that any two distinct Sylow p-subgroups intersect trivially. (Hint: Suppose to the contrary that there are two distinct Sylow p-subgroups that intersect non-trivially, and consider the normalizer of a largest such intersection.)

4. (a) (5 points) Prove that $\mathbb{Z}[\omega]$ is a Euclidean domain, where $\omega = \frac{1+\sqrt{-3}}{2}$.
   (b) (3 points) Prove that $\mathbb{Z}[\sqrt{-3}]$ is not a UFD.

5. Let $A = \{a_0 + a_3x^3 + a_4x^4 + \cdots + a_nx^n |\ n \in \mathbb{Z}^{\geq 0},\ a_i \in \mathbb{Z}\}$ be a subring of $\mathbb{Z}[x]$.
   (a) (2 points) Find the field fraction of A.
   (b) (5 points) Is A a UFD?
   (c) (3 points) Is there $f(x) \in \mathbb{Z}[x]$ such that A is generated as a ring by 1 and $f(x)$, i.e. $A = \mathbb{Z}[f(x)]$?

6. Let A be a unital commutative ring. Suppose that for some $n \in \mathbb{Z}^{\geq 0}$ we have

$$x^n = x$$

   for any $x \in A$.
   (a) (4 points) Prove that, for any prime idea $\mathfrak{p}$, $A/\mathfrak{p}$ is a field of order at most n.
   (b) (4 points) Prove that the intersection of all the maximal ideals of A is zero.
   (c) (4 points) Prove that, if A is Noetherian, then it is finite. (Hint:
      1. If $\mathfrak{a} \triangleleft A$, then

$$\mathrm{Ann}(\mathfrak{a}) := \{a \in A |\ a\mathfrak{a} = 0\} \triangleleft A,$$

      and, if $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathrm{Ann}(\mathfrak{b}) \subseteq \mathrm{Ann}(\mathfrak{a})$.
      2. Consider

$$\Sigma := \{\mathrm{Ann}(\cap_{i=1}^{n} \mathfrak{p}_i) |\ \mathfrak{p}_i \in \mathrm{Spec}(A)\},$$

      to prove that there are finitely many prime ideals $\mathfrak{p}_i$'s such that

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = 0.$$

      3. Deduce the A is finite.)