Math 109 Homework 3 Solutions

File A

4. We proceed by strong induction on n.

Base cases: For $n = 34, 35, 36, 37, 38$ it is possible to pay with stamps of denominations 5 and 9 as shown in the hint.

Inductive step: Let $n \geq 39$ and assume that it is possible to pay exactly k for any value of k such that $34 \leq k \leq n-1$. We will show that it is possible to pay n. Since $n \geq 39$, $34 \leq n-5 \leq n-1$, so it is possible to write $n - 5 = 5x + 9y$ for some nonnegative integers x and y. Adding 5 to both sides gives $n = 5(x+1) + 9y$, so it is possible to pay postage of value n.

By strong induction, it is possible to pay any value of postage greater than 34.

File B

1. (a)

$$\{A \in P(\{1,2,3,4\} \mid |A| \text{ is even}\} = \{\varnothing, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3,4\}\}$$

(b)

$$\{A \in P(\{1,2,3,4\} \mid |A| \text{ is odd}\} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$$

2. (a) True. The only positive real number whose square is 1 is 1, so

$$\{x \in \mathbb{R} \mid x > 0, (x^2-1)^2 = 0\} = \{1\}$$

so the set in question just has one element written two different ways, and its cardinality is indeed 1.

(b) False. Although $\{\varnothing\}$ is an *element* of $\{1, \{\varnothing\}\}$, it is not a subset of it. For it to be a subset, the empty set itself would need to be an element of $\{1, \{\varnothing\}\}$, and it is not.

(c) True. The set $\mathbb{R}$ is written in two different ways, so there are two elements.

3. (a) Let X be a set, and let A be an arbitrary subset of X. By the definition of symmetric difference, we have

$$x \in A\Delta\varnothing \Leftrightarrow ((x \in A) \wedge (x \notin \varnothing)) \vee ((x \notin A) \wedge (x \in \varnothing))$$

Since nothing is in the empty set, $x \in \varnothing$ is always false, while $x \notin \varnothing$ is always true. We can therefore simplify the above expression as follows:

$$((x \in A) \wedge (x \notin \varnothing)) \vee ((x \notin A) \wedge (x \in \varnothing)) \Leftrightarrow ((x \in A) \wedge \top) \vee ((x \notin A) \wedge \bot)$$

Since for any proposition P, $P \wedge \top \equiv P$ and $P \wedge \bot \equiv \bot$, this simplifies to

$$((x \in A) \wedge \top) \vee ((x \notin A) \wedge \bot) \Leftrightarrow (x \in A) \vee \bot \Leftrightarrow x \in A$$

Putting this all together gives

$$x \in A\Delta\varnothing \Leftrightarrow x \in A$$

as desired.

(b) Let X be a set, and let A be an arbitrary subset of X. By the definition of symmetric difference, we have

$$x \in A\Delta A \Leftrightarrow ((x \in A) \wedge (x \notin A)) \vee ((x \notin A) \wedge (x \in A))$$

Each half of the above disjunction is a contradiction, so this is always false. Since $x \in \varnothing$ is also always false, we have

$$x \in A\Delta A \Leftrightarrow \bot \Leftrightarrow x \in \varnothing$$

as desired.

(c) We can apply parts (a) and (b) as well as associativity per the hint to get

$$\begin{aligned} A\Delta B = A\Delta C &\Rightarrow A\Delta(A\Delta B) = A\Delta(A\Delta C) \\ &\Rightarrow (A\Delta A)\Delta B = (A\Delta A)\Delta C \\ &\Rightarrow \varnothing\Delta B = \varnothing\Delta C \\ &\Rightarrow B = C \end{aligned}$$

4. We proceed via proof by cases:

Case 1. $1 \in A$. Then

$$A\Delta\{1\} = A \setminus \{1\} \cup \{1\} \setminus A$$

Since $1 \in A$, $\{1\} \setminus A$ is the empty set, and we have

$$A \setminus \{1\} \cup \{1\} \setminus A = A \setminus \{1\} \cup \varnothing = A \setminus \{1\}$$

Since $1 \in A$, $|A \setminus \{1\}| = |A| - 1$, and $|A|$ is even if and only if $|A| - 1$ is odd.

Case 2. $1 \notin A$. Then

$$A\Delta\{1\} = A \setminus \{1\} \cup \{1\} \setminus A$$

Since $1 \notin A$, $\{1\} \setminus A = \{1\}$ and $A \setminus \{1\} = A$, so we have

$$A \setminus \{1\} \cup \{1\} \setminus A = A \cup \{1\}$$

Since $1 \notin A$, $|A \cup \{1\}| = |A| + 1$, and $|A|$ is even if and only if $|A| + 1$ is odd.

In either case, we have $|A|$ is even if and only if $|A\Delta\{1\}|$ is odd, so we are done by proof by cases.

5. (a) By the definition of $A \subseteq B$ we know $A \subseteq B$ is equivalent to

$$(x \in A) \Rightarrow (x \in B)$$

By the definition of $A \cap B$, we know that $A \cap B = A$ is equivalent to

$$((x \in A) \wedge (x \in B)) \Leftrightarrow (x \in A)$$

By checking truth tables, we can verify that $P \to Q \equiv (P \wedge Q) \Leftrightarrow P$. Applying this to $P = x \in A$ and $Q = x \in B$ proves that

$$A \subseteq B \Leftrightarrow A \cap B = A$$

(Note there are many ways to show $(x \in A) \Rightarrow (x \in B) \equiv ((x \in A) \wedge (x \in B)) \Leftrightarrow (x \in A)$, you do not need to use truth tables)

(b) Let X be a set, and let $A, B, C \subseteq X$. Assume that $A \cap B = A \cap C$ and $A \cup B = A \cup C$. We will show by cases that for all $x \in X$, $x \in B \Leftrightarrow x \in C$.

Case 1. $x \in A$. Then we have

$$x \in B \Leftrightarrow (x \in A) \wedge (x \in B) \Leftrightarrow x \in A \cap B \Leftrightarrow x \in A \cap C \Leftrightarrow (x \in A) \wedge (x \in C) \Leftrightarrow x \in C$$

Case 2. $x \notin A$. Then we have

$$\begin{aligned} x \notin B &\Leftrightarrow (x \notin B) \wedge (x \notin A) \\ &\Leftrightarrow \neg((x \in B) \vee (x \in A)) \\ &\Leftrightarrow x \notin (A \cup B) \\ &\Leftrightarrow x \notin (A \cup C) \\ &\Leftrightarrow \neg((x \in C) \vee (x \in A)) \\ &\Leftrightarrow (x \notin C) \wedge (x \notin A) \\ &\Leftrightarrow x \notin C \end{aligned}$$

Since the contrapositive is equivalent, we again have

$$x \in B \Leftrightarrow x \in C$$

By proof by cases we have

$$x \in B \Leftrightarrow x \in C$$

and

$$B = C$$

File C

1. (a) $\exists \epsilon > 0, \forall \delta > 0, (|x-1| < \delta) \wedge (|x^2 - 1| < \epsilon)$

(b) $\exists \epsilon > 0, \exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, |x - n| \geq \epsilon$.

(c) $\exists \epsilon > 0, \exists x \in \mathbb{R}, \forall m, n \in \mathbb{Z}, |x - m - n\alpha| \geq \epsilon$.

2. (a) Proof: Let $x = -2017$. Let $y \in \mathbb{R}$. Then $y^2 \geq 0 > -1 = 2016 + x$, so the statement holds.

(b) Disproof: Let $x \in \mathbb{R}$. Let $y = (2016+x)^{\frac{1}{3}}$ (note: it is important here that the exponent is $\frac{1}{3}$, to guarantee that this is well defined). Then

$$y^3 = 2016 + x,$$

and it is false that $y^3 > 2016 + x$.

(c) Proof: Let $\epsilon > 0$, and let N be an integer greater than $\frac{1000}{\epsilon}$. Let $n \geq N$. Then

$$\frac{1000}{n} \leq \frac{1000}{N} < \frac{1000}{\frac{1000}{\epsilon}} = \epsilon,$$

so the statement holds.