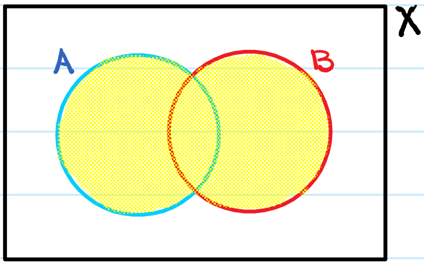
Lecture 08: Set operations

Thursday, October 20, 2016 4:37 PM

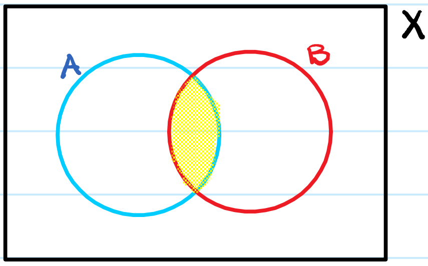
The union of A and B, $A \cup B$

$x \in A \cup B \Leftrightarrow x \in A \vee x \in B$.



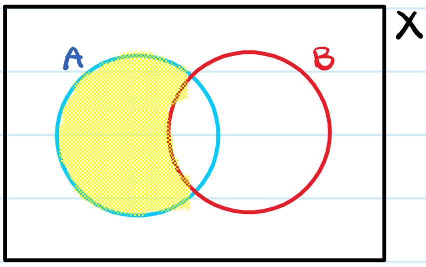
The intersection of A and B, $A \cap B$

$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B$.



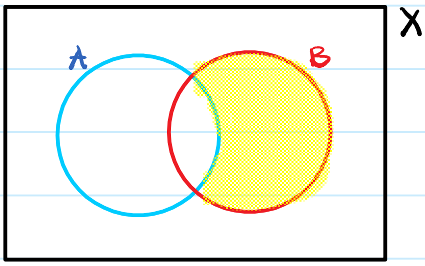
The difference set of A and B, $A \setminus B$

$x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B$.

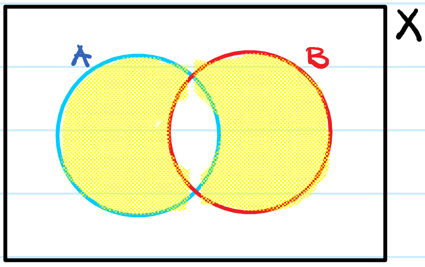


The difference set of B and A, $B \setminus A$

$x \in B \setminus A \Leftrightarrow x \in B \wedge x \notin A$.



The symmetric difference of A and B, $A \Delta B$



$A \Delta B = (A \setminus B) \cup (B \setminus A)$.

$x \in A \Delta B \Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$.

<u>Claim.</u> $A \Delta B = (A \cup B) \setminus (A \cap B)$.

We have to show that $x \in A \Delta B \Leftrightarrow x \in (A \cup B) \setminus (A \cap B)$.

Lecture 08: Properties of set operations

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We use a truth-table to show this.

$x\in A \vee x\in B$ $x\in A \wedge (\neg x\in B)$ $x\in A\setminus B \vee x\in B\setminus A$

$x\in A$	$x\in B$	$x\in A\cup B$	$x\in A\cap B$	$x\in (A\cup B)\setminus(A\cap B)$	$x\in A\setminus B$	$x\in B\setminus A$	$x\in A\triangle B$
T	T	T	T	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	F	T	F	T	T
F	F	F	F	F	F	F	F

$x\in A \wedge x\in B$ $x\in B \wedge (\neg x\in A)$

$x\in A\cup B \wedge (\neg\, x\in A\cap B)$

Hence, for any $x\in X$,

$$x\in (A\cup B)\setminus(A\cap B) \iff x\in A\triangle B,$$

which implies $(A\cup B)\setminus(A\cap B) = A\triangle B$. ∎

When we fix a "mother set" X, referred to as the universal set, we can talk about the complement of every subset A of X. The complement of A is $X\setminus A$, and it is denoted by A^c. Hence for all $x\in X$, $x\in A^c \Leftrightarrow x\notin A$.

There are certain parallels between propositional forms, sets, and numbers 0 and 1.

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$\cup$ $\leftrightsquigarrow$ $\vee$ $\leftrightsquigarrow$ max.
union disjunction maximum

$\cap$ $\leftrightsquigarrow$ $\wedge$ $\leftrightsquigarrow$ min.
intersection conjunction minimum

^{c} $\leftrightsquigarrow$ $\neg$ $\leftrightsquigarrow$ $x \mapsto 1-x$ (swapping 0 and 1).
Complement negation

Similar to the propositional forms, we have certain basic and extremely useful set equations.

<u>Properties of set operations.</u> Suppose X is a universal set, and $A, B, C \subseteq X$. Then

(1) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
(Associative)

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(Distributive)

(3) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$ (De Morgan's laws).

(4) $A \setminus B = A \cap B^c$

(5) $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$.

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(6) $A \subseteq B \overset{(a)}{\iff} A \cap B = A$

$\overset{(b)}{\iff} A \cup B = A$

$\overset{(c)}{\iff} A \setminus B = \emptyset$

(7) $A \cap B = \emptyset \iff A \subseteq B^c$.

<u>Proof</u>. For every $x \in X$,

(2) $x \in A \cup (B \cap C) \iff x \in A \vee x \in B \cap C$

$\iff x \in A \vee (x \in B \wedge x \in C)$

$\iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

$\iff x \in A \cup B \wedge x \in A \cup C$

$\iff x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. The other one is similar.

(3) $x \in (A \cup B)^c \iff x \notin A \cup B \iff \neg (x \in A \cup B)$

Remember that $x \in X$

$\iff \neg (x \in A \vee x \in B)$

$\iff \neg (x \in A) \wedge \neg (x \in B)$

$\iff x \in A^c \wedge x \in B^c$

$\iff x \in A^c \cap B^c$.

So $(A \cup B)^c = A^c \cap B^c$. The other one is similar.

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Remember $x \in X$

(4) $x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B \Leftrightarrow x \in A \wedge x \in B^c$

$\Leftrightarrow x \in A \cap B^c$

Therefore $A \setminus B = A \cap B^c$.

(5) $x \in A \cap B \Rightarrow x \in A \wedge x \in B$

$\Rightarrow x \in A$. Thus $A \cap B \subseteq A$.

$x \in A \Rightarrow x \in A \vee x \in B$

$\Rightarrow x \in A \cup B$. So $A \subseteq A \cup B$.

(6) $(\overset{(a)}{\Rightarrow})$ We have to show $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

The former is proved in (4).

$x \in A \Rightarrow x \in B$ since $A \subseteq B$. So $x \in A \Rightarrow (x \in A \wedge x \in B)$

$\Rightarrow x \in A \cap B$.

Hence $A \subseteq A \cap B$.

$(\overset{(a)}{\Leftarrow})$ By (4) we have $A \cap B \subseteq B$. By assumption

$A = A \cap B$. Hence $A \subseteq B$.

$(\overset{(b)}{\Rightarrow})$ We have to prove $B \subseteq A \cup B$ and $A \cup B \subseteq B$.

The former is proved in (4).

$x \in A \cup B \Rightarrow x \in A \vee x \in B$

Case 1. $x \in A \Rightarrow x \in B$ since $A \subseteq B$.

Case 2. $x \in B \Rightarrow x \in B$.

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So in either case we conclude $x \in B$. Therefore

$x \in A \cup B \Rightarrow x \in B$. So $A \cup B \subseteq B$.

($\overset{(b)}{\Leftarrow}$) By (4) we have $A \subseteq A \cup B$. By the assumption, $A \cup B = B$. Hence $A \subseteq B$.

($\overset{(c)}{\Rightarrow}$) We have to prove $A \subseteq B \Rightarrow A \setminus B = \emptyset$.

Suppose to the contrary for some subsets A and B we have $A \subseteq B \wedge A \setminus B \neq \emptyset$.

$$A \setminus B \neq \emptyset \Rightarrow \text{there exists } x_0 \in A \setminus B$$
$$\Rightarrow x_0 \in A \wedge x_0 \notin B$$
$$\Rightarrow x_0 \in B \wedge x_0 \notin B \quad \text{as } A \subseteq B.$$

This is a contradiction.

($\overset{(c)}{\Leftarrow}$) We have to show $A \setminus B = \emptyset \Rightarrow A \subseteq B$.

Suppose to the contrary there are subsets A and B such that

$A \setminus B = \emptyset \wedge A \not\subseteq B$.

So $\neg$(for $\underline{\underline{\text{any}}}$ x, $x \in A \Rightarrow x \in B$), which implies

for some x_0, $x_0 \in A \wedge x_0 \notin B$.

Lecture 08: Properties of set operations, quantifiers

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$\Rightarrow x_0 \in A\setminus B \Rightarrow A\setminus B \neq \emptyset$ which contradicts our assumption.

(6) Using part (3) and the fact that $(B^c)^c = B$, we have

$A \cap B = A \setminus B^c$. By part (5c) we have

$A \subseteq B^c \Leftrightarrow A\setminus B^c = \emptyset$.

Therefore $A \subseteq B^c \Leftrightarrow A \cap B = \emptyset$. ■

Quantifiers.

We have already discussed quanifiers and how they determine in what capacity a variable should be looked for within a set.

Universal quanifier: For all x in X, ... $\quad \forall x \in X, \ldots$

For every x in X, ...

Existential quanifier: There exists x in X, ... $\quad \exists x \in X, \ldots$

For some x in X, ...

Ex. To say 2 is prime is equivalent to

$$\forall\, m,n \in \mathbb{Z},\ 2 \mid mn \Rightarrow (2 \mid m \vee 2 \mid n).$$

Ex. The existential part of long division can be written as follows:

Lecture 08: Quantifiers

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$\forall a \in \mathbb{Z},\ \forall b \in \mathbb{Z} \setminus \{0\},\ \exists q, r \in \mathbb{Z},\ a = bq + r \wedge 0 \leq r < |b|$.

<u>Ex.</u> Write the well-ordering principle using mathematical language.

<u>Solution.</u> We do this in several steps.

$\underbrace{\text{Every}}_{\text{univeral quanifier}}$ non-empty $\underbrace{\text{subset}}_{A}$ of $\underbrace{\text{non-negative integers}}_{\mathbb{Z}^{\geq 0}}$ has a minimum.

$$\forall A \subseteq \mathbb{Z}^{\geq 0},\quad A \neq \emptyset \Rightarrow \quad A \text{ has a minimum}.$$

(Alternatively, $\forall\ A \in \mathcal{P}(\mathbb{Z}^{\geq 0}) \setminus \{\emptyset\} \Rightarrow A$ has a minimum.)

x is the minimum of A precisely when $x \in A$ and $x \leq y$ for every $y \in A$.

Hence A has a minimum exactly when $\exists x \in A, \forall y \in A,\ x \leq y$.

Altogether, the well-ordering principle is

$$\forall A \subseteq \mathbb{Z}^{\geq 0},\quad A \neq \emptyset \Rightarrow (\exists x \in A, \forall y \in A,\ x \leq y). \qquad \blacksquare$$

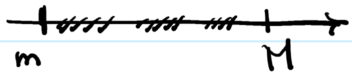
<u>Warning.</u> In general, one cannot switch the order of quantifiers. For instance, $\exists x \in A, \forall y \in A, x \leq y$ means A has a minimum, but $\forall y \in A, \exists x \in A, x \leq y$ is always true; for every $y \in A$, let $x = y$. Then $x \in A$ and $x \leq y$. In $\exists x \in A, \forall y \in A, \ldots$, x is supposed to work for all y, but in $\forall y \in A, \exists x \in A, \ldots$ x can depend on y.

Lecture 08: Quantifiers

Monday, October 24, 2016 9:22 AM

Ex. Use quantifiers to say a nonempty subset A of $\mathbb{R}$ is bounded.

Solution. Intuitively we say A is bounded if it is a subset of a finite length interval



$$\exists\, m, M \in \mathbb{R},\quad \forall\, a \in A,\quad m \leq a \leq M.$$

a lower bound an upper bound.

Notice m, M are NOT necessarily in A, and they are NOT unique.

Ex. Prove or disprove that every bounded non-empty subset $A \subseteq \mathbb{R}$ has a minimum.

Solution. We want to disprove it. To do so, it is necessary and sufficient to find a bounded non-empty subset with no minimum. We show that the open interval $(0,1)$ is bounded and it does not have a minimum.

Notice that, for every $a \in (0,1)$, $0 \leq a \leq 1$. Hence $(0,1)$ is bounded.

Next we show that $(0,1)$ does not have a minimum. Suppose to the contrary that x_0 is a minimum of $(0,1)$. (To be continued.)