Lecture 02: Conditional propositions

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In the previous lecture, we discussed equivalent propositional forms.

For instance, we showed that:

$$P \Rightarrow Q \equiv (\neg P) \vee Q,$$

(de Morgan's laws) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$.

$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q).$$

We discussed how to use "proof by contradiction" and "case-by-case proof" to show certain statement holds. We start by showing equivalence of certain propositional forms that justify why those methods are legitimate.

Proof by contradiction means in order to show P holds is equivalent to showing that $\neg P$ implies a contradiction. Let $\top$ denote a proposition which is always true (such a propositional is called a tautology), and $\perp$ denote a proposition which is always false (such a proposition is called a contradiction). So proof-by-contradiction means $P \equiv (\neg P) \Rightarrow \perp$.

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Ex. (Proof-by-contradiction) $P \equiv (\neg P) \Rightarrow \perp$.

Proof. $(\neg P) \Rightarrow \perp \equiv \neg(\neg P) \vee \perp$ $\quad H \Rightarrow C \equiv (\neg H) \vee C$

$\equiv P \vee \perp$

$\equiv P$

P	$\perp$	$P \vee \perp$
T	F	T
F	F	F

Ex. $P \wedge T \equiv P$.

$P \vee T \equiv T$.

$P \Rightarrow T \equiv T$.

$P \wedge \perp \equiv \perp$

Easier examples that I encourage you to check on your own.

Another important set of rules are distribution laws.

$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$, $\quad P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.

. Since most of mathematical statements are conditional propositions, we try to understand other propositional forms that are equivalent to $P \Rightarrow Q$.

. $P \Rightarrow Q \equiv (\neg P) \vee Q$ $\quad (1) \Rightarrow (2) \equiv (\neg 1) \vee (2)$

$\equiv Q \vee (\neg P)$ $\quad (1) \vee (2) \equiv (\neg(1)) \Rightarrow (2)$

$\equiv (\neg Q) \Rightarrow (\neg P)$

$(\neg Q) \Rightarrow (\neg P)$ is called the contra-positive of $P \Rightarrow Q$, we just showed that the contrapositive of $P \Rightarrow Q$ is equivalent to $P \Rightarrow Q$.

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Proof-by-contradiction in the context of conditional propositions works as follows: to show

$$\text{if } \underbrace{\text{hypothesis}}_{H}, \text{ then } \underbrace{\text{conclusion}}_{C}.$$

We can assume H holds and C does not hold and show that that implies a contradiction. In the language of propositional forms, we are claiming,

$$H \Rightarrow C \equiv (H \wedge (\neg C)) \Rightarrow \perp$$

<u>Ex.</u> $P \Rightarrow Q \equiv (P \wedge (\neg Q)) \Rightarrow \perp$.

<u>Proof</u>. $P \Rightarrow Q \equiv (\neg P) \vee Q$ (1)

(We use a combination of forward and backward arguments)

We maniuplate both sides to end up getting propositional forms that involve only $\neg$, $\vee$, $\wedge$.

$$\begin{aligned}(P \wedge (\neg Q)) \Rightarrow \perp &\equiv \neg(P \wedge (\neg Q)) \vee \perp \\ &\equiv \neg(P \wedge (\neg Q)) \\ &\equiv (\neg P) \vee \neg(\neg Q) && \text{(de Morgan's laws)} \\ &\equiv (\neg P) \vee Q && (2)\end{aligned}$$

By (1) and (2), we deduce that $P \Rightarrow Q \equiv (P \wedge (\neg Q)) \Rightarrow \perp$.

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Last time we discussed case-by-case proof. This means hypothesis has two (or more cases) and we want to reach to a conclusion. To do so we can consider each case separately. In the language of propositional forms, it can be written as follows:

$$\underbrace{(P_1 \vee P_2)}_{\text{case 1 or case 2}} \Rightarrow \underbrace{Q}_{\text{conclusion}} \equiv \underbrace{(P_1 \Rightarrow Q) \wedge (P_2 \Rightarrow Q)}_{\text{consider each case separately}}$$

<u>Proof.</u> Again we work with both sides and find equivalent propositional forms that involve only $\neg, \vee, \wedge$.

$$(P_1 \vee P_2) \Rightarrow Q \equiv \neg(P_1 \vee P_2) \vee Q$$

$$\equiv ((\neg P_1) \wedge (\neg P_2)) \vee Q \qquad \text{(de Morgan's law)}$$

$$\equiv ((\neg P_1) \vee Q) \wedge ((\neg P_2) \vee Q) \quad (1) \quad \text{(distribution)}$$

$$(P_1 \Rightarrow Q) \wedge (P_2 \Rightarrow Q) \equiv ((\neg P_1) \vee Q) \wedge ((\neg P_2) \vee Q) \quad (2)$$

By (1) and (2), we obtain that

$$(P_1 \vee P_2) \Rightarrow Q \equiv (P_1 \Rightarrow Q) \wedge (P_2 \Rightarrow Q). \qquad \blacksquare$$

Lecture 02: Case-by-case proof

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As an example, suppose you are solving a 3x3 suduko type of puzzle and you are given the following table

		1
2		

.

You want to argue why the (1,1) box is 3.

You say, suppose to the contrary that the (1,1) box is not 3. Then it is either 1 or 2. Say P_1 is the proposition which says "the (1,1) box is 1" and P_2 is the proposition which says "the (1,1) box is 2". You want to show that

$$(P_1 \vee P_2) \Rightarrow \perp .$$

To do so, you consider each case separately, and discuss why

$$(P_1 \Rightarrow \perp) \quad \wedge \quad (P_2 \Rightarrow \perp) .$$

$P_1 \Rightarrow \perp$ because there is another box with 1 in the first row.

$P_2 \Rightarrow \perp$ because there is another box with 2 in the first column.

- In math, we often have to show that given a hypothesis, we might have <u>two</u> possible outcomes. In the language of propositional forms, this means $P \Rightarrow (Q_1 \vee Q_2)$.

The following table can help us visualize our possible methods

hyp.	goal
P	Q
$P \wedge (\neg Q)$	$\perp$

Proof-by-contradiction

hyp.	goal
P	Q
$\neg Q$	$\neg P$

Contrapositive

hyp.	goal
P	$Q_1 \vee Q_2$
$P \wedge (\neg Q_1)$	Q_2
$P \wedge (\neg Q_1) \wedge (\neg Q_2)$	$\perp$

In the language of propositional forms, we have to show the following.

Ex $P \Rightarrow (Q_1 \vee Q_2) \equiv (P \wedge (\neg Q_1)) \Rightarrow Q_2$

$$\equiv (P \wedge (\neg Q_1) \wedge (\neg Q_2)) \Rightarrow \perp .$$

Proof. We start with each one of these propositional forms and find an equivalent form which involves only $\neg, \vee, \wedge$.

- $P \Rightarrow (Q_1 \vee Q_2) \equiv (\neg P) \vee (Q_1 \vee Q_2) \equiv (\neg P) \vee Q_1 \vee Q_2$ ...... (1)

- $(P \wedge (\neg Q_1)) \Rightarrow Q_2 \equiv \neg (P \wedge (\neg Q_1)) \vee Q_2$

$$\equiv ((\neg P) \vee \neg(\neg Q_1)) \vee Q_2 \quad \text{(de Morgan's law)}$$

$$\equiv (\neg P) \vee Q_1 \vee Q_2 \quad \text{.......... (2)}$$

- $(P \wedge (\neg Q_1) \wedge (\neg Q_2)) \Rightarrow \perp \equiv \neg (P \wedge (\neg Q_1) \wedge (\neg Q_2)) \vee \perp$

(de Morgan's law) $\equiv (\neg P) \vee \neg(\neg Q_1) \vee \neg(\neg Q_2)$

$$\equiv (\neg P) \vee Q_1 \vee Q_2 \quad \text{.......... (3)}$$

Lecture 02: Case-by-case proof

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By (1), (2), and (3), we conclude that

$$P \Rightarrow (Q_1 \vee Q_2) \equiv (P \wedge (\neg Q_1)) \Rightarrow Q_2 \equiv (P \wedge (\neg Q_1) \wedge (\neg Q_2)) \Rightarrow \bot .$$

■

<u>Ex.</u> For every real numbers a and b,

$$ab \neq 0 \Leftrightarrow (a \neq 0 \text{ and } b \neq 0).$$

($P \Leftrightarrow Q$ is called <u>biconditional proposition</u>. This means $P \Rightarrow Q$ and $Q \Rightarrow P$. We read it in one of the following ways:
- P if and only if Q,
- P exactly when Q,
- P precisely when Q,
- P is necessary and sufficient for Q.)

<u>Proof.</u> We have to show two implications:

$$(\underbrace{ab \neq 0}_{P} \Rightarrow (\underbrace{a \neq 0}_{Q_1} \text{ and } \underbrace{b \neq 0}_{Q_2})) \text{ and } ((a \neq 0 \text{ and } b \neq 0) \Rightarrow ab \neq 0)$$

To show the first one, we show its contrapositive holds:

$\neg(Q_1 \wedge Q_2) \Rightarrow \neg P$. This means we have to show
$((\neg Q_1) \vee (\neg Q_2)) \Rightarrow (\neg P)$

$((a=0) \text{ or } (b=0)) \Rightarrow (ab=0)$. We use a case-by-case proof. <u>Case 1.</u> If $a=0$, then $ab=(0)(b)=0$. <u>Case 2.</u> If $b=0$, then $ab=(a)(0)=0$. Show the 2nd part using a similar method. ■

(You case use the fact that $ab=0$ implies $a=0$ or $b=0$ for $a,b \in \mathbb{R}$.)

Lecture 02: Inequality and backward argument

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Proving an inequality can often be hard. Next we prove a basic inequality using a "backward argument".

Theorem. For every real numbers x and y, $x^2+y^2 \geq 2xy$.

for all (universal quantifier)

$$\forall\ x,y \in \mathbb{R}, \quad x^2+y^2 \geq 2xy$$

in $\quad$ set of real numbers

Proof / draft.

hypothesis	goal
x, y real	$x^2+y^2 \geq 2xy$
$\Downarrow$	$\Uparrow$
$x-y$ real	$x^2+y^2-2xy \geq 0$
	$\Uparrow$
	$(x-y)^2 \geq 0$

Finding simpler inequalities that imply our goal.

(In the previous lecture, we proved for every real number a, $a^2 \geq 0$.)

In math books and articles, statements are labelled by Lemma, Proposition, Theorem. All are true propositions. Lemmas are often auxiliary statements. Propositions are considered "mini" theorems. In math books, we often build upon our previous results. Here is an example.

Theorem. For every real numbers x, y, z, $x^2+y^2+z^2 \geq xy+xz+yz$.

$$\forall\ x,y,z \in \mathbb{R}, \quad x^2+y^2+z^2 \geq xy+xz+yz.$$

Lecture 02: Inequality and Constructing proofs

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This time we "construct" a proof using our previous Theorem.

<u>Proof.</u> By the previous theorem, we obtain that

$$x^2+y^2 \geq 2xy, \quad x^2+z^2 \geq 2xz, \text{ and } \quad y^2+z^2 \geq 2yz.$$

Adding these inequalities, we deduce that

$$2(x^2+y^2+z^2) \geq 2(xy+xz+yz).$$ Therefore,

$$x^2+y^2+z^2 \geq xy+xz+yz. \qquad \blacksquare$$

We use basic arithmetics to practice various methods of proving theorems. In general, the following concepts are the key points that one obtains from a mathematical education:

(1) Problem solving.

(2) Pattern recognition.

(3) Abstract thinking.

These concepts are well-presented in all the aspects of this course, and you should have them in mind when you go over the materials.

Lecture 02: Divisibility

Wednesday, September 28, 2016 9:42 AM

Definition. Suppose m and n are two integers. We say

<u>m divides n</u> if, for some integer k,

$$n = mk.$$

In this case we also say <u>n is a multiple of m</u>. And it is denoted by $m \mid n$.

Basic Properties of divisibility.

- For any integer n, $1 \mid n$.

  Proof. Since $n = 1 \times n$ and n is an integer, n is a multiple of 1. ■

- For any integer n, $n \mid 0$.

  Proof. Since $0 = n \times 0$ and 0 is an integer, 0 is a multiple of n. ■

Lemma. Suppose a and b are integers.

$$(b \neq 0 \land a \mid b) \Longrightarrow |a| \leq |b|.$$

Proof. Since $a \mid b$, for some integer k we have
$$b = ak.$$

Lecture 02: Divisibility and backward argument

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<u>Claim</u> $k \neq 0$.

<u>Proof of claim</u>. Suppose to the contrary that $k = 0$.

Then $b = ak = a \times 0 = 0$, which contradics our assumption that $b \neq 0$. ■

Since $k \neq 0$, $|k| > 0$. Since k is integer, we have $|k| \geq 1$. Hence $|k|\,|a| \geq |a|$.

Therefore $|a| \leq |k|\,|a| = |ka| = |b|$. ■

How did we know that we need to show $k \neq 0$?

Whenever you want to prove an implication, it is useful to write down what your hypothesis is and what your goal is. Then start moving <u>forward in the hypothesis</u> side and <u>backward in the goal</u> side.

Lecture 02: Backward argument

Wednesday, September 28, 2016 10:53 PM

Hypothesis

$b \neq 0 \wedge a \mid b$

$\Downarrow$

$b \neq 0 \wedge b = ak$ for some integer k

forward.

?

Goal

$|a| \le |b|$

$\Uparrow$

$|a| \le |ak|$

$\Uparrow$

$|a| \le |a||k|$

$\Uparrow$

$1 \le |k|$

$\Uparrow$

$k \neq 0 \wedge k \in \mathbb{Z}$

backward.

(recall that $0<1$ implies $1<2$, $2<3$, and so on; and similarly $0>-1$, $-1>-2$, ... Hence $|k|$ is one of the numbers $0<1<2<\cdots$. Therefore, if $k\neq 0$, then $|k|\ge 1$.)