Lecture 1: Proposition

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In this course you learn

- How to listen to a proof and understand it.
- How to read a proof and understand it.
- How to produce a proof and communicate your thoughts.

We will use different parts of mathematics to achieve this goal. We start with Propositional Logic, introduce quantifiers, use basic ideas from game theory, discuss ε-δ definition of limit, study a little bit of arithmetic. The key to success, however, is doing lots of exercises.

Mathematical Language

Proposition is a sentence that is either true or false (not at the same time!).

Ex. $1+1$ is NOT a proposition. A proposition has to claim something. That claim might be true or false. A sentence with no claim is not a proposition.

Ex. $1+1=3$ is a proposition. It is a false proposition.

Ex. $m=1$. It is NOT a proposition. We do not know where m lives and in what capacity should we look for it.

Lecture 1: Propositional logic

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We refer to the "universe" where a free variable "lives" as a set, and the "capacity" is determined by quanifiers. For instance "for all ..." or "for every ..." is called the universal quantifier and it is denoted by $\forall$

All $\rightsquigarrow$ A $\rightsquigarrow$ $\forall$

"For all real number x, ..." is written as $\forall x \in \mathbb{R}$,

in the set of real numbers.

Existential quantifier is used when we want to say that "There exists ..." or "There is ..." or "For some ...".

It is denoted by $\exists$

Exists $\rightsquigarrow$ E $\rightsquigarrow$ $\exists$

"There exists a real number x ..." is written as $\exists x \in \mathbb{R}$.

There are two other less standard quantifiers:

"There is no ..." which is denoted by $\nexists$ and "There is a unique..." which is denoted by $\exists!$. Later we will discuss sets and quanifiers, more.

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Using sets and quantifiers, we can make the predicate ce similar $m=1$ into a proposition. We refer to m as the free variable of this predicate.

Ex. For every rational number m, $m=1$. ($\forall m \in \mathbb{Q}, m=1$.)

This is a proposition. In fact, this is a false proposition.

To see it is false, it is enough to present a counter-example.

This means find a rational number m which is not 1.

Since this proprosition claims that certain property should hold for every rational number, finding a single rational number which does NOT satisfy the claimed property shows that this is a false proposition.

2 is a rational number and $2 \neq 1$. So 2 is a counter-example.

Ex. If m is integer and $0 < m < 2$, then $m=1$.

This is a true proposition.

To show this is true, we have to recall axioms of inequalities.

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By axioms, we mean certain statements that we assume are true, and infer other statements from them. This is the common approach in math, and it dates back to the work of Euclid on geometry. In this course, we take the following properties of inequality as axioms.

It should be said that some of these statements can be proved based on others.

- For every real numbers a, b, c, d, the following statement holds.

(i) $a<b$ and $b<c$ imply $a<c$

(ii) $a<b$ or $a=b$ or $b<a$

(iii) $a<b$ implies $a+c<b+c$

(iv) $a<b$ and $c<d$ imply $a+c<b+d$

(v) $a<b$ and $0<c$ imply $ac<bc$

(vi) $a<b$ and $c<0$ imply $ac>bc$

both of these can be deduced from $0<a$ and $0<b$ imply $0<ab$

(vii) $0<1$.

Using (vii) and (iv), we deduce $0+1<1+1$; and so $1<2$.

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By adding 1 again and again, we conclude that $0<1<2<\cdots$.

This is not very precise. When we have a process that needs to be repeated, we must often use induction. This will be discussed later in the course.

By a similar argument and adding -1 to the both sides of $0<1$ many times, we obtain that $\cdots<-2<-1<0$.

As we can see, the only integer which is more than 0 and less than 2 is 1.

Ex. $x^2 \geq 0$

It is NOT a proposition. This is a predicate with a free variable x. Using a quantifier and a set to which the free variable x belongs, we can turn this predicate into a proposition.

For instance:

Ex. For every real number x, $x^2 \geq 0$. $(\forall x \in \mathbb{R}, x^2 \geq 0.)$.

This is a true proposition. Let's prove why this is true.

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We use a case-by-case proof.

Case 1. $x>0$. Then multiplying both sides by the positive number x, we conclude that $x\cdot x > x\cdot 0$, and so $x^2>0$.

(In algebra, you will learn that is 0 is the neutral element of $+$; that mean $x+0=0+x=x$, we have cancelation with respect to $+$; that means $x+y=x+z$ implies $y=z$, and the distribution property then $x\cdot 0=0$ for every x. Here is a proof: $0+0=0$ implies $x\cdot(0+0)=x\cdot 0$, and so $x\cdot 0+x\cdot 0=x\cdot 0$. Therefore by the cancelation property $x\cdot 0=0$. In this course, you can use this property without proof.)

Case 2. $x=0$. Then $x^2=0$, and so $x^2\geq 0$.

Case 3. $x<0$. Multiplying both sides by the negative number x, $x\cdot x > x\cdot 0$, and so $x^2>0$. ∎

We use either Q.E.D. or ∎ to indicate that proof is finished.

Question. Can we avoid using the axiom on multiplying by a negative number?

Adding $(-x)$ to the both sides of $x<0$ implies that

$$(-x)+x<(-x)+0; \text{ and so } 0<-x.$$

Multiplying the positive number by the both sides of $x<0$, we infer that $(-x)\cdot x<(-x)\cdot 0$; and so $(-x)\cdot x<0$.

Notice that $(-x)\cdot x+x\cdot x=((-x)+x)\cdot x=0\cdot x=0$, and so $(-x)\cdot x=-x^2$ (during the lecture, we took this equality for granted.)

Hence, $-x^2<0$. Adding x^2 to the both sides of $(-x^2)<0$, we conclude that $0<x^2$. ∎

Lecture 1: Russell's example.

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Ex. This is a false sentence.

. It is NOT a proposition.

Suppose to the contrary that it is a proposition.

(This method of proof is called proof by contradiction.)

Then there are two possibilities:

(This method of arguing is called case-by-case proof.)

Case 1. It is a true proposition.	Case 2. It is a false proposition.
So the claim of this proposition is supposed to be true, which says it is false. That is a contradiction.	So the claim of this proposition is supposed to be false, which implies that it is true. That is a contradiction.

The only reason that we are getting a contradiction is because we assumed that the above sentence is a proposition. Hence this is NOT a valid assumption, i.e. the above sentence is NOT a proposition.

Lecture 1: Propositional connectives

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As you can see in the previous examples, we often have to work with various propositions that are connected to each other by certain propositional connectives. To understand these connectives better, we are going to use variables for various propositions.

Conjunction. The conjunction of two propositions P, Q is "P and Q", and it is denoted by $P \wedge Q$.

As you can imagine, $P \wedge Q$ is true precisely when both P and Q are true. We can use a truth table and list all the possible cases for the truth values of the pair of propositions P and Q.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

True exactly when P is T and Q is T.

Disjunction. The disjunction of two propositions P, Q is "P or Q", and it is denoted by $P \vee Q$.

Disjunction is slightly different from the way we use "or" in our

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daily conversations. To understand the truth value of $P \vee Q$ better, think about when a claim $P \vee Q$ is false. When I make a claim that "either P holds or Q" and you want to argue that my claim is false, you have show that (there is a scenario such that) neither P nor Q holds. That means both P and Q are false. Hence $P \vee Q$ is False exactly when both P and Q are false.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Here if both P and Q are true, then $P \vee Q$ is true. This is slightly different from the daily usage of "or". Here, if $P \wedge Q$ holds, then so does $P \vee Q$.

Conditional proposition or implication. It is denoted by $P \Rightarrow Q$ we refer to P as the hypothesis of this conditional proposition, and to Q as the conclusion of this conditional proposition.

We read it as:

Lecture 1: Propositional connectives

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- If P, then Q.
- P implies Q.
- P is sufficient for Q.
- Q is necessary for P.

To understand the truth value of $P \Rightarrow Q$, let's think when an implication can fail. When I claim "P implies Q" and you want to argue that my claim is false, you need to come up with a scenario where P holds (hypothesis is true) and Q fails (conclusion is false). Hence, "P implies Q" is false exactly when P is true and Q is false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The only case where $P \Rightarrow Q$ is false.

If P is false, then $P \Rightarrow Q$ is true regardless of the truth-value of Q. A false hypothesis can imply anything!

Lecture 1: Propositional forms and truth tables

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A propositional form is a legitimate expression involving logical variables, connectives $\wedge, \vee, \Rightarrow, \Leftrightarrow, \neg$, and (,).
(will be discussed later)

Here are the truth table of conjunction $\wedge$, disjunction $\vee$, and implication $\Rightarrow$. (Summary)

P and Q, $P \wedge Q$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

It is "and" so both should hold for P and Q to hold

all the possibilities of the truth values of P, Q.
. If there are 3 varibles, the number of rows is 8.
For 4 variables, we get 16 rows.
For n variables, we get 2^n rows.

P or Q, $P \vee Q$

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

For this to fail, both P and Q should fail

This is slightly different from the way "or" is used in a daily language. There we sometimes assume P and Q do not hold at the same time.

P implies Q, $P \Rightarrow Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication fails only when hypothesis (P) holds and conclusion (Q) fails.

In particular $P \Rightarrow Q$ is true if P is false.

Lecture 1: Equivalence of propositional forms

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Ex. $(1=2) \Rightarrow$ Sun is a moon.

is a true proposition as the hypothesis is false.

Of course I am NOT claiming that the conclusion "sun is a moon" is correct. We are only saying that the implication is true.

Ex. Write the truth table of $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Definition Two propositional forms are called equivalent if they have the same truth tables.

Ex. $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$

Similarly $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$

} de Morgan's law.

Convention. $\top$: a true proposition.

$\perp$: a false proposition. (contradiction).

Lecture 1: An equivalent form of an implication

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Ex. Show that $P \Rightarrow Q \equiv (\neg P) \vee Q$.

Proof. We are going to write the truth tables of these propositional forms and compare their values.

P	Q	$\neg P$	$(\neg P) \vee Q$	$P \Rightarrow Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	F

So the propositional forms $P \Rightarrow Q$ and $(\neg P) \vee Q$ always have the same truth-values. Hence, $P \Rightarrow Q \equiv (\neg P) \vee Q$.

This is an extremely important fact. Using this to show "P implies Q" is the same as arguing that "either P is false or Q is true."