HOMEWORK 7 SOLUTIONS

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Problem 1

(a) We know that $\mathbb{1}_A(x)$ will contribute 1 to the sum each time $x \in A$, and 0 otherwise, so $\sum_{x\in X} \mathbb{1}_A(x) = |A|$.

(b) Summing both sides over $x \in X$ and applying part (a) gives

$$\begin{aligned}
|A \cup B| &= \sum_{x\in X} \mathbb{1}_{A\cup B}(x) \\
&= \sum_{x\in X} \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x) \\
&= \sum_{x\in X} \mathbb{1}_A(x) + \sum_{x\in X} \mathbb{1}_B(x) - \sum_{x\in X} \mathbb{1}_{A\cap B}(x) \\
&= |A| + |B| - |A \cap B|
\end{aligned}$$

(c) When $n = 1$ we have $\mathbb{1}_{A_1^c} = (1 - \mathbb{1}_{A_1})$ by 3(b) from HW6, since $\mathbb{1}_X$ is the constant function 1. Assume the claim holds for some integer $n \geq 1$. Then we have

$$\begin{aligned}
\mathbb{1}_{(A_1\cup\cdots\cup A_{n+1})^c} &= \mathbb{1}_{(A_1\cup\cdots A_n)^c\cap A_{n+1}^c} \\
&= \mathbb{1}_{(A_1\cup\cdots A_n)^c}\mathbb{1}_{A_{n+1}^c} \\
&= (1 - \mathbb{1}_{A_1})\cdots(1 - \mathbb{1}_{A_n})(1 - \mathbb{1}_{A_{n+1}}).
\end{aligned}$$

By induction, the claim holds for all $n \geq 1$.

(d) Again, we'll proceed by induction. When $n = 1$ we have $\mathbb{1}_{A_1^c} = 1 - \mathbb{1}_{A_1}$ which agrees with the stated formula. Now suppose the claim holds for some integer $n \geq 1$. Then we have

$$\begin{aligned}
\mathbb{1}_{(A_1\cup\cdots\cup A_{n+1})^c} &= \mathbb{1}_{(A_1\cup\cdots A_n)^c\cap A_{n+1}^c} \\
&= \mathbb{1}_{(A_1\cup\cdots A_n)^c}\mathbb{1}_{A_{n+1}^c} \\
&= \left(1 + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}}\right)(1 - \mathbb{1}_{A_{n+1}}) \\
&= \left(1 + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}}\right) - \mathbb{1}_{A_{n+1}}\left(1 + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}}\right) \\
&= \left(1 + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}}\right) - \left(\mathbb{1}_{A_{n+1}} + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}\cap A_{n+1}}\right)
\end{aligned}$$

Notice how the first double sum gives us terms where the subscripts consist of intersections not involving A_{n+1} and the second double sum gives us terms which do involve A_{n+1}. Furthermore, the minus sign in front of the second double sum ensures that when a subscript involves a $k+1$ term intersection, the sign in front of it is $(-1)^{k+1}$. Thus, we may combine terms to obtiain:

$$\left(1 + \sum_{k=1}^{n+1}(-1)^k \sum_{i_1<\cdots<i_k} \mathbb{1}_{A_{i_1}\cap\cdots\cap A_{i_k}}\right)$$

as desired. Summing both sides over $x \in X$ and applying part (a) gives the inclusion-exclusion formula.

Problem 2

(a) By associativity of function composition we have:

$$g = g \circ I_y = g \circ (f \circ h) = (g \circ f) \circ h = I_X \circ h = h$$

so we conclude that $g = h$.

(b) Since f is a bijection it is injective and surjective. According to your class notes, this implies that f has a left inverse g and a right inverse h. By part (a), $g = h$. Thus, g is a candidate for the inverse of f since it is both a left and right inverse. To show uniqueness, suppose g' is also a left and right inverse of f. Since g is a left inverse and g' is a right inverse, part (a) implies that $g = g'$. Therefore the inverse is unique.

Problem 3

(a) Suppose $g \circ f(x) = g \circ f(y)$ for some $x, y \in X$. Since g is injective, this implies $f(x) = f(y)$. Since f is injective, $x = y$. Therefore $g \circ f$ is injective. Now let $c \in Z$ be arbitrary. Since g is surjective, there exists $b \in Y$ such that $g(b) = c$. Since f is surjective, there exists $a \in X$ such that $f(a) = b$. Then we have $g \circ f(a) = g(b) = c$, so $g \circ f$ is surjective. Since $g \circ f$ is injective and surjective, it is a bijection.

(b) Suppose $f^{-1}(x) = f^{-1}(y)$ for some $x, y \in Y$. Applying f to both sides we have $f \circ f^{-1}(x) = f \circ f^{-1}(y)$, which is equivalent to saying $I_Y(x) = I_Y(y)$. Thus, $x = y$, so f^{-1} is injective. Now let $a \in X$ be arbitrary and set $b = f(a)$. Then $b \in Y$ and we have $f^{-1}(b) = f^{-1}(f(a)) = I_X(a) = a$, so f^{-1} is surjective. Therefore f^{-1} is bijective.

Problem 4

First suppose g is injective. Let $b \in Y$ be arbitrary. Let $c = g(b)$. Since $g \circ f$ is surjective, there exists $a \in X$ such that $g \circ f(a) = c$. Since $g(f(a)) = c = g(b)$ and g is injective, we have $f(a) = b$. Therefore f is surjective.

Now suppose f is surjective. Suppose $g(y_1) = g(y_2)$ for some $y_1, y_2 \in Y$. Since f is surjective, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Thus, $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is injective, this implies $x_1 = x_2$. Applying f to both sides, we have $f(x_1) = f(x_2)$ so $y_1 = y_2$. Therefore g is injective.

Problem 5

(a) If f is bijective then it has a unique inverse by 2(b), which is also a left inverse. 2(a) implies that it is unique, since the inverse is also a right inverse. Now suppose f has a unique left inverse. (Note: the reverse implication is actually only true when $|X| > 1$. We'll see why in a minute.) The fact that f has a left inverse implies f is injective, so we need to show f is surjective. Suppose toward a contradiction that f fails to be surjective. Then there exists some $y \in Y$ such that $y \neq f(x)$ for any $x \in X$. Let g be a left inverse of f and define a function $h : Y \to X$ such that $h(b) = g(b)$ for all $b \in Y$ such that $b \neq y$, and $h(y)$ is any element in x which is not equal to $g(y)$. Such an element must exist since $|X| > 1$. Then h is also a left inverse of f because for all $x \in X$ we have $h(f(x)) = g(f(x)) = x$. (This relies on the fact that $y \neq f(x)$ for any $x \in X$.) This contradicts uniqueness of g. Therefore f is a bijection.

(b) If f is bijective then it has a unique inverse by 2(b), which is also a right inverse. 2(a) implies that it is unique, since the inverse is also a left inverse. Now suppose f has a unique right inverse. Since f has a right inverse this implies f is surjective, so we just need to show f is injective. Suppose toward a contradiction that f fails to be injective. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. Let g be a right inverse of f and define $h : Y \to X$ such that $h(b) = g(b)$ for all $b \in Y$ such that $b \neq f(x_1)$. Define $h(f(x_1))$ to be x_1 if $g(f(x_1)) \neq x_1$, and $h(f(x_1)) = x_2$ if $g(f(x_1)) \neq x_2$. Then for all $y \neq f(x_1)$ we have $f(h(y)) = f(g(y))$. We also have $f(h(f(x_1)) = f(x_1) = f(g(f(x_1))$, so h is in fact a right inverse of f. Finally, $h(f(x_1)) \neq g(f(x_1))$, so h and g are not the same function. This contradicts uniqueness of g. Therefore f must be injective, so we conclude f is a bijection.