# HOMEWORK 6 SOLUTIONS

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## Problem 1

$$\begin{aligned}(x,y) \in A \times (B \cup C) &\iff x \in A \wedge y \in B \cup C \\ &\iff x \in A \wedge (y \in B \vee y \in C) \\ &\iff (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\ &\iff (x,y) \in A \times B \vee (x,y) \in A \times C \\ &\iff (x,y) \in A \times B \cup A \times C.\end{aligned}$$

Therefore the two sets are equal.

## Problem 2

(a) This statement is true. Suppose toward a contradiction that there exists some $x \in \mathbb{R}$ such that for all $\varepsilon > 0$ we have $|x| < \varepsilon$ and $x \neq 0$. Setting $\varepsilon = |x|/2$ implies that $|x| < |x|/2$,which is impossible. Thus, the statement is true.

(b) This statement is false. To see this, set $x = 1$ and $\varepsilon = 2$. Then $|x| < \varepsilon$ but it is not the case that $x = 0$.

## Problem 3

(a) To show the functions are the same, it suffices to check that they give the same output for any input. Let $x \in X$. If $x \in A \cap B$ then $\mathbb{1}_{A\cap B} = 1$. On the other hand, $x \in A$ and $x \in B$, so we have $\mathbb{1}_A(x) \cdot \mathbb{1}_B(x) = 1 \cdot 1 = 1$.

If $x \notin A \cap B$ then $\mathbb{1}_{A\cap B}(x) = 0$. On the other hand, if $x \notin A \cap B$ then either $x \notin A$ or $x \notin B$ so at least one of the functions $\mathbb{1}_A(x)$ and $\mathbb{1}_B(x)$ evaluates to 0, making $\mathbb{1}_A(x) \cdot \mathbb{1}_B(x) = 0$.

(b) Let $x \in X$. If $x \in A$ then

$$\mathbb{1}_A(x) + \mathbb{1}_{A^c}(x) = 1 + 0 = 1 = \mathbb{1}_X(x).$$

If $x \notin A$ then

$$\mathbb{1}_A(x) + \mathbb{1}_{A^c}(x) = 0 + 1 = 1 = \mathbb{1}_X(x).$$

(c) Let $x \in X$. We'll first consider the case where $x \in A \cup B$, so that $\mathbb{1}_{A\cup B}(x) = 1$. If $x \in A \cap B$ then by part (a) we have

$$\mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_A(x)\mathbb{1}_B(x) = \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x) = 1 + 1 - 1 = 1.$$

If $x \notin A \cap B$ then $x$ is in exactly one of $A$ or $B$, so we have

$$\mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_A(x)\mathbb{1}_B(x) = \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x) = 1 + 0 - 0 = 1.$$

Now we'll consider the case where $x \notin A \cup B$. Then $\mathbb{1}_{A\cup B}(x) = 0$ and we have

$$\mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_A(x)\mathbb{1}_B(x) = \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x) = 0 + 0 - 0 = 0.$$

Therefore the functions are the same.

(d)

$$\begin{aligned}
\mathbb{1}_{A\setminus B} &= \mathbb{1}_A \cdot \mathbb{1}_{B^c} && \text{by (a)}\\
&= \mathbb{1}_A(\mathbb{1}_X - \mathbb{1}_B) && \text{by (b)}\\
&= \mathbb{1}_A \cdot \mathbb{1}_X - \mathbb{1}_A \cdot \mathbb{1}_B \\
&= \mathbb{1}_{A\cap X} - \mathbb{1}_{A\cap B} && \text{by (a)}\\
&= \mathbb{1}_A - \mathbb{1}_{A\cap B}
\end{aligned}$$

(e)

$$\begin{aligned}
\mathbb{1}_{A\Delta B} &= \mathbb{1}_{A\cup B\setminus A\cap B} \\
&= \mathbb{1}_{A\cup B} - \mathbb{1}_{A\cup B} \cdot \mathbb{1}_{A\cap B} && \text{by (d)}\\
&= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} - \mathbb{1}_{(A\cup B)\cap(A\cap B)} && \text{by (c) and (a)}\\
&= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} - \mathbb{1}_{A\cap B} && \text{since } A\cap B \subset A \cup B\\
&= \mathbb{1}_A + \mathbb{1}_B - 2\mathbb{1}_A \cdot \mathbb{1}_B
\end{aligned}$$

(f) Let $A, B \subset X$. To prove the forward direction of the implication, suppose that $\forall x \in X$ we have that $\mathbb{1}_A(x) \le \mathbb{1}_B(x)$. Now let $a \in A$. Then we have $\mathbb{1}_A(a) = 1 \le \mathbb{1}_B(a)$. Since $\mathbb{1}_B$ can only take values in $\{0, 1\}$ this implies $\mathbb{1}_B(a) = 1$. In other words, that $x \in B$. Thus, $A \subset B$.

For the reverse direction of the implication, suppose $A \subset B$. Then for any $x \in A$ we have $\mathbb{1}_A(x) = 1$. Since $A \subset B$ we also have $x \in B$, so $\mathbb{1}_B(x) = 1$. In particular, $\mathbb{1}_A(x) \le \mathbb{1}_B(x)$. If $x \notin A$ then $0 = \mathbb{1}_A(x) \le \mathbb{1}_B(x)$ since any characterstic function is at least 0.

If $A = B$ then clearly $\mathbb{1}_A = \mathbb{1}B$. On the other hand, if $\mathbb{1}_A = \mathbb{1}_B$ then $\mathbb{1}_A \le \mathbb{1}_B$ and $\mathbb{1}_B \le \mathbb{1}_A$. This implies $A \subset B$ and $B \subset A$, so $A = B$.

**Problem 4**

Suppose $\Theta(A) = \Theta(B)$ for some $A, B \in P(X)$. Then $\mathbb{1}_A = \mathbb{1}_B$. By problem 3f, this implies $A = B$. Therefore $\Theta$ is injective. Next, let $f \in F(X, \{0, 1\})$ be arbitrary. We'll define a set $A$ as follows:

$$A = \{a \in X | f(a) = 1\}.$$

Then we have $\Theta(A) = \mathbb{1}_A$. Furthermore, for any $x \in X$ we have

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} = \begin{cases} 1 & f(x) = 1 \\ 0 & f(x) = 0 \end{cases} = f(x).$$

Therefore $\Theta$ is surjective. Since $\Theta$ is injective and surjective, it is a bijection.