# HOMEWORK 5 SOLUTIONS

MICHELLE BODNAR

**Problem 1**

(a)
$$\exists \varepsilon > 0, \forall \delta > 0, |x-1| < \delta \wedge |x^2 - 1| \geq \varepsilon$$

(b)
$$\exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, |x - n| \geq \varepsilon$$

(c) Let $\alpha$ be an irrational number.
$$\exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall m, n \in \mathbb{Z}, |x - m - n\alpha| \geq \varepsilon$$

**Problem 2**

(a) This statement is true. Let $x = -2017$. Then for any real number $y$ we have $y^2 \geq 0 > -1 = 2016 + x$.

(b) This statement is false. Suppose towards a contradiction that there were such an $x$. Then we can set $y = \sqrt[3]{2016 + x}$, so we have $y^3 = 2016 + x$, which contradicts the fact that $y^3 > 2016 + x$.

(c) This statement is true. Let $\varepsilon > 0$ be arbitrary. By the hint, we know there exists an integer which is strictly greater than $1000/\varepsilon$. Let $N$ be such an integer. Then for any $n \geq N$ we have
$$\frac{1000}{n} \leq \frac{1000}{N} < \varepsilon.$$

**Problem 3**

Suppose toward a contradiction that there exist $L_1, L_2 \in \mathbb{R}$ such that for all $\varepsilon > 0$ we have $|L_1 - L_2| < \varepsilon$ but $L_1 \neq L_2$. Let $\varepsilon = \frac{|L_1 - L_2|}{2}$. (Note: since $L_1 \neq L_2$, we know that $\varepsilon$ is in fact strictly greater than 0, so this is a valid choice of $\varepsilon$.) Then we have $|L_1 - L_2| < \frac{|L_1 - L_2|}{2}$ which is impossible. □

**Problem 4**

(a)
$$\forall \varepsilon > 0, \exists N \in \mathbb{Z}_{>0}, \forall n \in \mathbb{Z}, n \geq N \implies |x_n - a| < \varepsilon.$$

(b) Suppose toward a contradiction that the limit does exist. In other words, there exists $L \in \mathbb{R}$ such that $\lim_{x \to a} f(x) = L$. Before doing any actual work, let's think how you might approach a problem like this. Since we assumed the limit exists, we know that when the inputs to $f$ are close to $a$, the outputs are close to $L$. We know that the $x_n^+$ terms get close to $a$, and when we input them to $f$ we get outputs close to $L_1$. This should tell you that $L_1$ must be close to $L$. On the other hand, the $x_n^-$ terms get close to $a$ as well, and when we input them to $f$ we get outputs close to $L_2$. This means that $L_2$ must be close to $L$. In particular, both $L_1$ and $L_2$ can be made as close to $L$ as we like, but $L_1 \neq L_2$, so at some point this must fail. That "failure" is exactly what will get us our contradiction. To make all this precise, we need to go back to definitions.

Let's get started by writing down all the inequalities we get for free from our hypotheses. I'll number them so we may refer to them easily later. Let $\varepsilon > 0$. By the definition of limit, there exists some $\delta > 0$ such that $|x - a| < \delta$ implies

$$|f(x) - L| < \frac{\varepsilon}{4}. \tag{1}$$

Since $x_n^+ \to a$, there exists $N_1$ such that $n \geq N_1$ implies

$$|x_n^+ - a| < \delta. \tag{2}$$

(Note: Here, $\delta$ is playing the role of $\varepsilon$ from our definition in part (a)).

Since $x_n^- \to a$, there exists $N_2$ such that $n \geq N_2$ implies

$$|x_n^- - a| < \delta. \tag{3}$$

Since $f(x_n^+) \to L_1$, there exists $N_3$ such that $n \geq N_3$ implies

$$|f(x_n^+) - L_1| < \frac{\varepsilon}{4}. \tag{4}$$

Since $f(x_n^-) \to L_2$, there exists $N_4$ such that $n \geq N_4$ implies

$$|f(x_n^-) - L_2| < \frac{\varepsilon}{4}. \tag{5}$$

Let $N = \max(N_1, N_2, N_3, N_4)$. Then for $n \geq N$, inequalities (1), (2), (3), and (4) will all simultaneously hold. In particular, for any $n \geq N$ we have:

$$\begin{aligned}
|L_1 - L_2| &= |L_1 - L + L - L_2| \\
&\leq |L_1 - L| + |L - L_2| \text{ by the triangle inquality } (|x+y| \leq |x| + |y| \text{ for any } x, y \in \mathbb{R}) \\
&= |L_1 - f(x_n^+) + f(x_n^+) - L| + |L - f(x_n^-) + f(x_n^-) - L_2| \\
&\leq |L_1 - f(x_n^+)| + |f(x_n^+) - L| + |L - f(x_n^-)| + |f(x_n^-) - L_2| \text{ by triangle inequality}
\end{aligned}$$

We can now bound each of these terms with ease: By (4), $|L_1 - f(x_n^+)| < \varepsilon/4$. By (5), $|f(x_n^-) - L_2| < \varepsilon/4$. By (1) and (2), we know $|x_n^+ - a| < \delta$ and $|x_n^- - a| < \delta$. Thus, by (1) we have that $|f(x_n^+) - L| < \varepsilon/4$ and $|L - f(x_n^-)| < \varepsilon/4$. Putting this all together, we have

$$|L_1 - L_2| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Since this argument works for any $\varepsilon > 0$, problem 3 tells us that $L_1 = L_2$, which contradicts our assumption that they were distinct. $\square$