Lecture 21: Injection and having a left inverse

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In the previous lecture we defined an invertible function:

We say $X \xrightarrow{f} Y$ is invertible if

(1) it has a left inverse: $\exists\ Y \xrightarrow{g} X,\ g\circ f = I_X$

(2) it has a right inverse: $\exists\ Y \xrightarrow{h} X,\ f\circ h = I_Y$

Theorem. Suppose $X \xrightarrow{f} Y$ is a function.

f is invertible if and only if f is bijective.

We start by proving two lemmas:

Lemma 1. Suppose $X \xrightarrow{f} Y$ is a function.

f has a left inverse if and only if f is injective.

Proof. ($\Rightarrow$) $\exists\ Y \xrightarrow{g} X,\ g\circ f = I_X$ which is injective.

Hence, by a theorem that we proved in the previous lecture, f is injective.

($\Leftarrow$) Suppose $X \xrightarrow{f} Y$ is injective. We would like to define a function $Y \xrightarrow{g} X$ such that $g\circ f = I_X$, which means, for any $x \in X$, $g(f(x)) = x$.

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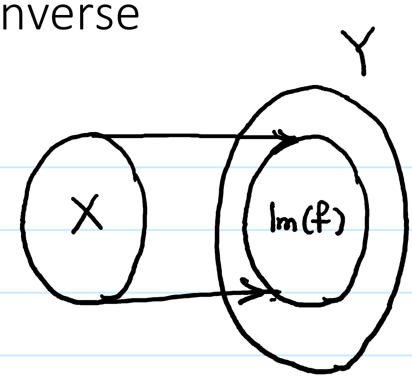
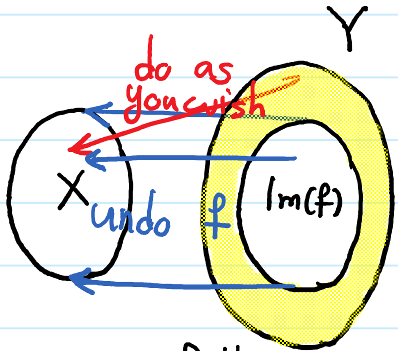


This means g should undo f on the image of f and can be anything outside of $\mathrm{Im}(f)$.



Here is a formal definition:

Choose $x_0 \in X$ (we can do that since $X \neq \emptyset$). Define $Y \xrightarrow{g} X$ as follows:

$$g(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \in X \\ x_0 & \text{if } y \in Y \setminus \mathrm{Im}(f). \end{cases}$$

We need to show g is a function (we say g is well-defined).

[Recall that to show "an assigning rule" defines a function from X to Y, we have to check three things:

1. This "rule" can be applied to all the elements of X.
2. This "rule" assigns elements of Y to any element of X.
3. This "rule" assigns a unique element of Y to any element of X.

For instance, we have seen that $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = y$ if $y^2 = x$ does NOT define a function. This rule assigns two elements of $\mathbb{R}$ to 1. Both 1 and -1 are assigned to 1.]

And then we have to check that $g \circ f = I_X$.

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Well-definedness of g It clearly assigns elements of Y to any element of X. We have to check why it assigns a unique element:

- If $y \in Y \setminus \mathrm{Im}(f)$, then x_0 is assigned to y with no ambiguity.
- Suppose $y \in \mathrm{Im}(f)$, and x_1 and x_2 can be assigned to y. So $f(x_1) = y \wedge f(x_2) = y$, which implies $f(x_1) = f(x_2)$. Since f is injective and $f(x_1) = f(x_2)$, we get that $x_1 = x_2$. So a unique element of X is assigned to y.

Checking $g \circ f = I_X$.

Both $g \circ f$ and I_X are functions from X to X. So we have to check only that $(g \circ f)(x) = I_X(x)$ for any $x \in X$.

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(y) && \text{where } y = f(x) \\ &= x && \text{the way we defined } g. \\ &= I_X(x). \end{aligned}$$

$\blacksquare$

Lecture 21: Surjection and having a left inverse

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Lemma 2. Suppose $f: X \to Y$ is a function.

f has a right inverse if and only if f is surjective.

In the proof we will be using an axiom of set theory called axiom of choice. First proof will be written and then it will be mentioned where axiom of choice is used.

Proof. ($\Rightarrow$) $\exists\, h: Y \to X$, $f \circ h = I_Y$. Since I_Y is surjective, f is surjective. (In the previous lecture we have proved that $f_1 \circ f_2$ is surjective implies that f_1 is surjective.)

($\Leftarrow$) We assume f is surjective. And we have to find $h: Y \to X$ such that $(f \circ h)(y) = y$. So h should be defined in a way such that $f(h(y)) = y$.

For any $y \in Y$, let $\overleftarrow{f}(y) = \{x \in X \mid f(x) = y\}$ be the preimage of y. Since f is surjective, $\overleftarrow{f}(y) \neq \emptyset$ for any $y \in Y$. Let's choose one element of $\overleftarrow{f}(y)$ and call it $h(y)$. So we get a function $h: Y \to X$ such that $h(y) \in \overleftarrow{f}(y)$.

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So $f(h(y)) = y$. Hence $f\circ h = I_Y$ as both of these functions are from Y to Y and $(f\circ h)(y) = f(h(y)) = y = I_Y(y)$.

(almost) $\square$

For a <u>single</u> non-empty set Z, we can get $z \in Z$. But to do it simultaneously for a <u>family</u> of non-empty sets, one needs <u>axiom of choice</u>:

> Suppose $F: Y \to P(X)$ be a function such that
>
> $\forall y \in Y,\ F(y) \neq \emptyset$.
>
> Then there is a function $h: Y \to X$ such that
>
> $\forall y \in Y,\ h(y) \in F(y)$.

Using the axiom of choice for $F: Y \to P(X)$, $F(y) = \overleftarrow{f}(y)$ we get the desired $h: Y \to X$.

$\blacksquare$

Lecture 21: Bijection and being invertible

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<u>Proof of Theorem</u>. (f is invertible $\Leftrightarrow$ f is bijective.)

f is invertible $\Rightarrow \begin{cases} f \text{ has a left inverse} \Rightarrow f \text{ is injective (by Lemma 1).} \\ f \text{ has a right inverse} \Rightarrow f \text{ is surjective (by Lemma 2).} \end{cases}$

f is injective and surjective $\Rightarrow$ f is bijective.

($\Leftarrow$) f is bijective $\Rightarrow$ f is injective and surjective

$\left.\begin{array}{l} f \text{ is injective} \Rightarrow f \text{ has a left inverse (by Lemma 1).} \\ f \text{ is surjective} \Rightarrow f \text{ has a right inverse (by Lemma 2).} \end{array}\right\} \Rightarrow f \text{ is invertible.}$

$\blacksquare$