

Lecture 18: Identity function

Wednesday, November 2, 2016 9:19 AM

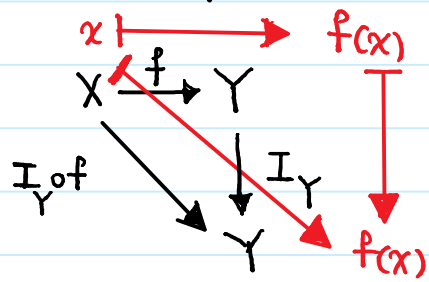
For any non-empty set X , the identity function I_X of X is

$$I_X: X \rightarrow X, I_X(x) = x \text{ for any } x \in X.$$

Lemma For any function $f: X \rightarrow Y$, we have

$$I_Y \circ f = f = f \circ I_X.$$

Proof.



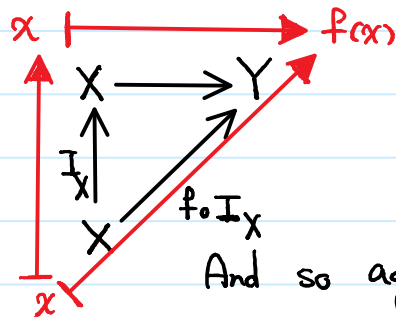
$$f: X \rightarrow Y \quad x \mapsto f(x)$$

$$I_Y \circ f: X \rightarrow Y$$

$$x \mapsto I_Y(f(x)) = f(x).$$

So they have the same (co)domain and rules. Hence

they are equal functions.

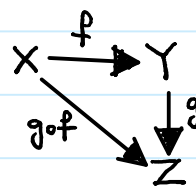


$$(f \circ I_X)(x) = f(I_X(x)) = f(x).$$

And so again $f \circ I_X$ and f have the same

(co)domain and rules. Thus $f \circ I_X = f$. ■

Remark. We say this diagram commutes: it does NOT matter which path we choose.



Lecture 18: Example of composite functions

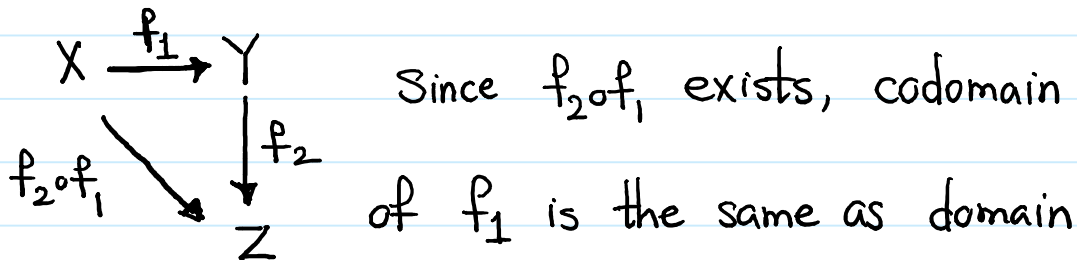
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Ex. Complete the missing information, if any.

$$f_1(x) = x+1, \quad f_2(x) = \sqrt{x}, \quad \text{and}$$

$$f_2 \circ f_1: \mathbb{R}^{\geq 0} \rightarrow \mathbb{R}, \quad f_2 \circ f_1(x) = \sqrt{x+1}.$$

Solution. Domains and co-domains of f_1 and f_2 are missing.



of f_2 . Let's denote it by Y .

$$\text{domain of } f_1 = \text{domain of } f_2 \circ f_1 = \mathbb{R}^{\geq 0}.$$

$$\text{codomain of } f_2 = \text{codomain of } f_2 \circ f_1 = \mathbb{R}.$$

• So $f_1: \mathbb{R}^{\geq 0} \rightarrow Y$, $f_1(x) = x+1$. In particular,

$$\forall x \in \mathbb{R}^{\geq 0}, \quad f_1(x) = x+1 \in Y. \text{ So}$$

$$x \in \mathbb{R}^{\geq 1} \Rightarrow x \in Y \text{ and so } \mathbb{R}^{\geq 1} \subseteq Y.$$

• $f_2: Y \rightarrow \mathbb{R}$, $f_2(y) = \sqrt{y}$ in order to get a function which is defined at every $y \in Y$, we should assume

$$y \in Y \Rightarrow y \geq 0 \Rightarrow y \in \mathbb{R}^{\geq 0}. \text{ Thus } Y \subseteq \mathbb{R}^{\geq 0}.$$

Hence Y can be any set $\mathbb{R}^{\geq 1} \subseteq Y \subseteq \mathbb{R}^{\geq 0}$. ■

Lecture 18: Image of a function

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Definition. Let $X \xrightarrow{f} Y$. Image of f is a subset of codomain:

$$\text{Im}(f) = \{f(x) \mid x \in X\}.$$

So we have $\forall y \in Y, (y \in \text{Im}(f) \iff \exists x \in X, y = f(x))$.

Definition. A function $X \xrightarrow{f} Y$ is called surjective or onto

if $\text{Im}(f) = Y$.

So we have

$$X \xrightarrow{f} Y \text{ is surjective} \iff \forall y \in Y, \exists x \in X, y = f(x)$$

In another words, for any $y \in Y$, you can solve the equation

$y = f(x)$ for $x \in X$.

Ex. Let $f: \mathbb{R}^{\geq 2} \rightarrow \mathbb{R}$, $f(x) = x^3$. Find $\text{Im}(f)$.

(we will use the facts that $x \mapsto x^3$ and $x \mapsto \sqrt[3]{x}$ are increasing functions.)

Solution. We claim $\text{Im}(f) = \mathbb{R}^{\geq 8}$. We need to show $\text{Im}(f) \subseteq \mathbb{R}^{\geq 8}$

and $\mathbb{R}^{\geq 8} \subseteq \text{Im}(f)$.

$\text{Im}(f) \subseteq \mathbb{R}^{\geq 8}$. To show this we have to verify

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$$y \in \text{Im}(f) \stackrel{?}{\Rightarrow} y \in \mathbb{R}^{\geq 8}.$$

$$y \in \text{Im}(f) \Rightarrow \exists x \in \mathbb{R}^{\geq 2}, y = f(x) = x^3.$$

Since $x \mapsto x^3$ is increasing and $x \geq 2$, we have

$$x^3 \geq 8. \text{ So } y \geq 8, \text{ which means } y \in \mathbb{R}^{\geq 8}.$$

$\mathbb{R}^{\geq 8} \subseteq \text{Im}(f)$. We have to show

$$y \in \mathbb{R}^{\geq 8} \Rightarrow y \in \text{Im}(f),$$

$$\text{which means } \forall y \in \mathbb{R}^{\geq 8}, \exists x \in \mathbb{R}^{\geq 2}, y = x^3.$$

$$\text{If } y \geq 8, \text{ then } x = \sqrt[3]{y} \geq \sqrt[3]{8} = 2 \text{ and } y = x^3$$

$$\text{So for any } y \in \mathbb{R}^{\geq 8}, y = (\sqrt[3]{y})^3 \text{ and } \sqrt[3]{y} \in \mathbb{R}^{\geq 2}. \quad \blacksquare$$

Definition. Graph of $X \xrightarrow{f} Y$ is a subset of $X \times Y$:

$$G_f = \{ (x, f(x)) \mid x \in X \}.$$

We will see several examples in the next lecture.