Lecture 18: Identity function

Wednesday, November 2, 2016 9:19 AM

For any non-empty set X, the identity function I_X of X is

$I_X : X \to X$, $I_X(x) = x$ for any $x \in X$.

<u>Lemma</u> For any function $f : X \to Y$, we have

$$I_Y \circ f = f = f \circ I_X .$$

<u>Proof.</u>

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "I_Y \circ f"'] & Y \arrow[d, "I_Y"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[dr, mapsto] & f(x) \arrow[d, mapsto] \\ & f(x) \end{tikzcd}$$

$f : X \to Y$ $\quad x \mapsto f(x)$

$I_Y \circ f : X \to Y$

$x \mapsto I_Y(f(x)) = f(x)$.

So they have the same (co)domain and rules. Hence they are equal functions.

$$\begin{tikzcd} & X \arrow[r] & Y \\ & X \arrow[u, "I_X"] \arrow[ur, "f \circ I_X"'] & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] & f(x) \\ x \arrow[u, mapsto] \arrow[ur, mapsto] & \end{tikzcd}$$

$(f \circ I_X)(x) = f(I_X(x))$

$= f(x)$.

And so again $f \circ I_X$ and f have the same (co)domain and rules. Thus $f \circ I_X = f$. ■

<u>Remark</u>. We say this diagram commutes: it does NOT matter which path we choose.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Lecture 18: Example of composite functions

Thursday, November 3, 2016 1:14 AM

Ex. Complete the missing information, if any.

$$f_1(x) = x+1, \quad f_2(x) = \sqrt{x}, \quad \text{and}$$

$$f_2 \circ f_1 : \mathbb{R}^{\geq 0} \longrightarrow \mathbb{R}, \quad f_2 \circ f_1(x) = \sqrt{x+1}.$$

Solution. Domains and co-domains of f_1 and f_2 are missing.

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[dr, "f_2 \circ f_1"'] & Y \arrow[d, "f_2"] \\ & Z \end{tikzcd}$$

Since $f_2 \circ f_1$ exists, codomain of f_1 is the same as domain of f_2. Let's denote it by Y.

domain of f_1 = domain of $f_2 \circ f_1 = \mathbb{R}^{\geq 0}$.

codomain of f_2 = codomain of $f_2 \circ f_1 = \mathbb{R}$.

- So $f_1 : \mathbb{R}^{\geq 0} \longrightarrow Y$, $f_1(x) = x+1$. In particular, $\forall x \in \mathbb{R}^{\geq 0}$, $f_1(x) = x+1 \in Y$. So

$$x \in \mathbb{R}^{\geq 1} \Rightarrow x \in Y \text{ and so } \mathbb{R}^{\geq 1} \subseteq Y.$$

- $f_2 : Y \longrightarrow \mathbb{R}$, $f_2(y) = \sqrt{y}$ in order to get a function which is defined at every $y \in Y$, we should assume

$$y \in Y \Rightarrow y \geq 0 \Rightarrow y \in \mathbb{R}^{\geq 0}. \quad \text{Thus } Y \subseteq \mathbb{R}^{\geq 0}.$$

Hence Y can be any set $\mathbb{R}^{\geq 1} \subseteq Y \subseteq \mathbb{R}^{\geq 0}$. ■

Lecture 18: Image of a function

Thursday, November 3, 2016 8:27 AM

Definition. Let $X \xrightarrow{f} Y$. Image of f is a subset of codomain:

$$\mathrm{Im}(f) = \{f(x) \mid x \in X\}.$$

So we have $\boxed{\forall y \in Y, \ (y \in \mathrm{Im}(f) \Leftrightarrow \exists x \in X, \ y = f(x)).}$

Definition. A function $X \xrightarrow{f} Y$ is called surjective or onto if $\mathrm{Im}(f) = Y$.

So we have

$$\boxed{X \xrightarrow{f} Y \text{ is surjective } \Leftrightarrow \forall y \in Y, \exists x \in X, \ y = f(x)}$$

In another words, for any $y \in Y$, you can solve the equation $y = f(x)$ for $x \in X$.

Ex. Let $f: \mathbb{R}^{\geq 2} \to \mathbb{R}$, $f(x) = x^3$. Find $\mathrm{Im}(f)$.

(we will use the facts that $x \mapsto x^3$ and $x \mapsto \sqrt[3]{x}$ are increasing functions.)

Solution. We claim $\mathrm{Im}(f) = \mathbb{R}^{\geq 8}$. We need to show $\mathrm{Im}(f) \subseteq \mathbb{R}^{\geq 8}$ and $\mathbb{R}^{\geq 8} \subseteq \mathrm{Im}(f)$.

$\mathrm{Im}(f) \subseteq \mathbb{R}^{\geq 8}$. To show this we have to verify

Lecture 18: Image of a function

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$y \in \mathrm{Im}(f) \overset{?}{\Rightarrow} y \in \mathbb{R}^{\geq 8}$.

$y \in \mathrm{Im}(f) \Rightarrow \exists\, x \in \mathbb{R}^{\geq 2},\ y = f(x) = x^3$.

Since $x \mapsto x^3$ is increasing and $x \geq 2$, we have $x^3 \geq 8$. So $y \geq 8$, which means $y \in \mathbb{R}^{\geq 8}$.

$\mathbb{R}^{\geq 8} \subseteq \mathrm{Im}(f)$. We have to show

$$y \in \mathbb{R}^{\geq 8} \Rightarrow y \in \mathrm{Im}(f),$$

which means $\forall y \in \mathbb{R}^{\geq 8}, \exists\, x \in \mathbb{R}^{\geq 2},\ y = x^3$.

If $y \geq 8$, then $x = \sqrt[3]{y} \geq \sqrt[3]{8} = 2$ and $y = x^3$

So for any $y \in \mathbb{R}^{\geq 8}$, $y = (\sqrt[3]{y})^3$ and $\sqrt[3]{y} \in \mathbb{R}^{\geq 2}$. $\blacksquare$

<u>Definition</u>. Graph of $X \xrightarrow{f} Y$ is a subset of $X \times Y$:

$$G_f = \{(x, f(x)) \mid x \in X\}.$$

We will see several examples in the next lecture.