Lecture 17: Cartesian product

Monday, October 31, 2016 9:17 AM

René Descarte used coordinates to study geometry. Nowadays we use the idea of n-tuples in many aspects of our life:

Ex. List of courses: it has various columns; name, number, location, ...

List of movies in netflix: genre, title, length, rating, etc.

Definition. Given sets X and Y, the Cartesian product of X and Y, denoted by $X \times Y$, is the set

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\},$$

where (x,y) is an ordered-pair, i.e. $(x_1,y_1)=(x_2,y_2)$ exactly when $x_1=x_2$ and $y_1=y_2$.

Similarly we define $X_1 \times X_2 \times \cdots \times X_n = \{(x_1,\dots,x_n) \mid x_i \in X_i \text{ for } 1 \le i \le n\}$, and $(x_1,\dots,x_n)=(x_1',\dots,x_n')$ if and only if $x_i=x_i'$ for $1 \le i \le n$.

Ex. Let $A=\{1,2\}$ and $B=\{a,b\}$. List elements of $A\times B$, and $B \times A$.

Solution. $A\times B=\{(1,a),(1,b),(2,a),(2,b)\}$

$B\times A=\{(a,1),(a,2),(b,1),(b,2)\}$

Lecture 17: Cartesian product

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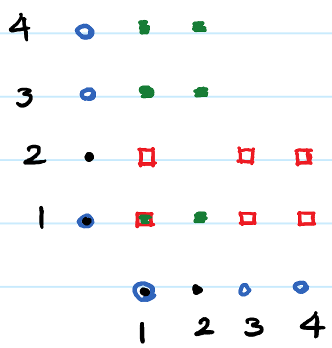
We pair each element of A by all the elements of B.

In the above example, you can see that $(A\times B)\cap(B\times A)=\emptyset$.

<u>Ex.</u> Let $A=\{1,2\}$ and $B=\{1,3,4\}$. Find $(A\times B)\cap(B\times A)$.

<u>Solution</u>



$$A\times B=\{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4)\}$$

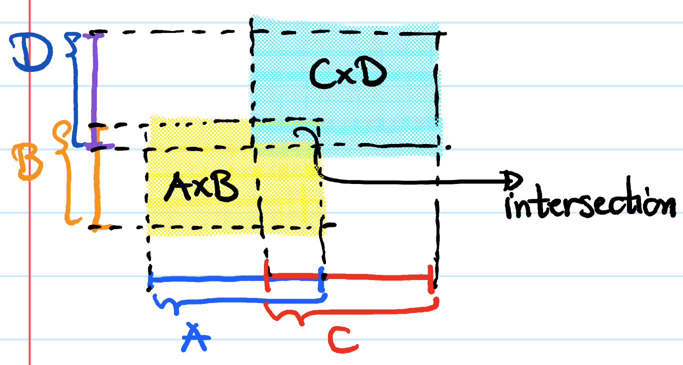
$$B\times A=\{(1,1),(1,2),(3,1),(3,2),(4,1),(4,2)\}$$

$$(A\times B)\cap(B\times A)=\{(1,1)\}. \quad \blacksquare$$

<u>Lemma</u>. $(A\times B)\cap(C\times D)=(A\cap C)\times(B\cap D)$.

<u>Proof</u>.

$$(x,y)\in(A\times B)\cap(C\times D)\iff(x,y)\in A\times B\wedge(x,y)\in C\times D$$



$$\iff x\in A\wedge y\in B\wedge x\in C\wedge y\in D$$

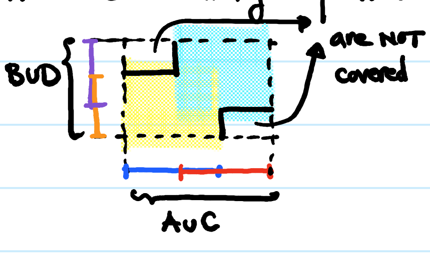
$$\iff(x\in A\wedge x\in C)\wedge(y\in B\wedge y\in D)$$

$$\iff x\in A\cap C\wedge y\in B\cap D$$

$$\iff(x,y)\in(A\cap C)\times(B\cap D).\ \blacksquare$$

<u>Warning</u>. $(A\times B)\cup(C\times D)$ is not necessarily equal to $(A\cup C)\times(B\cup D)$.

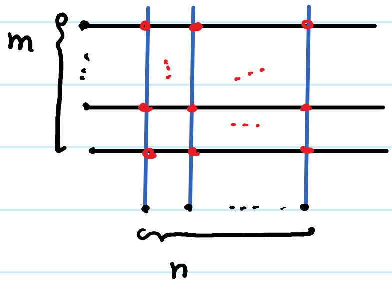
(why ?)



Lecture 17: Cartesian product and counting

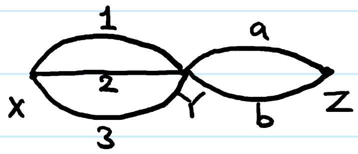
Tuesday, November 1, 2016 8:39 AM

Based on your intuition of cardinality of finite sets, you can see that $|A \times B| = |A|\,|B|$ if A and B are finite sets.



Ex. In the following pictures in how many ways can we go from X to Z by passing Y only once.

Solution. We can "label" each path with an element of $\{1,2,3\} \times \{a,b\}$. And any element of $\{1,2,3\} \times \{a,b\}$ is a label of a path. So there is a "matching" (the technical term is bijection as we will learn later) between the possible paths and elements of $\{1,2,3\} \times \{a,b\}$



So there are 6 possible paths. ∎

The key point in the above example is the following:

We often count objects by finding a bijection between them and a more familiar set. A set whose cardinality is already known.

Lecture 17: Functions

Tuesday, November 1, 2016 8:53 AM

"Definition" A function carries three pieces of information:

- Two sets: one is called domain and the other is called codomain.
- A rule: assigns a unique element of codomain to each element of domain

We either write $f: X \to Y$ and then specify its rule,

or

$$\begin{aligned} X &\xrightarrow{f} Y \\ x &\longmapsto f(x) \end{aligned}$$

- You have worked with a lot of functions in calculus, but in an inaccurate way. In the following examples we will see some of these inaccuracies.

Ex. Is the following a function?

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = 1/x.$$

Answer. No, f is NOT defined 0. ■

By changing its domain, we can address this issue:

$$f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f(x) = 1/x \quad \text{is a function.}$$

Lecture 17: Function, composition

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Ex. Is the following a function?

$$f: \mathbb{R} \to \mathbb{R}^+, \quad f(x) = x^2.$$

Answer. No, it is NOT. It assigns 0 to 0 which does NOT belong to the codomain $\mathbb{R}^+$. ∎

By changing the codomain we can address this issue:

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2 \quad \text{is a function.}$$

Ex. Is the following a function?

$$f: \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = y \text{ if } y^2 = x.$$

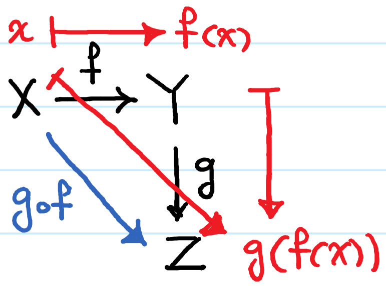
Answer. No, it is NOT. This rule does NOT assign a unique element of codomain to, let's say, 1. We have $(\pm 1)^2 = 1$. ∎

Changing the codomain can resolve this issue:

$$\left(f: \mathbb{R}^+ \to \mathbb{R}^+, \ f(x) = y \text{ if } y^2 = x \right) \text{ is a function.}$$

In fact, in this case, $f: \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = \sqrt{x}$.

Composition of functions Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be two functions; suppose codomain of f is equal to the domain of g. Then we can form a new function called the composition of f and g, denoted by $g \circ f$.



$(g \circ f)(x) = g(f(x))$

Domain of $g \circ f$ = Domain of f

Codomain of $g \circ f$ = codomain of g

Rule of $g \circ f$: $x \mapsto g(f(x))$.

<u>Ex.</u> Let $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $f(x) = 1/x$. Find $f \circ f$.

<u>Answer.</u> It does NOT make sense to talk about $f \circ f$ a codomain of f is NOT equal to the domain of f. ■

This issue can be resolved by changing the codomain of f.

Let $f: \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $f(x) = 1/x$. Then

$f \circ f : \mathbb{R}\setminus\{0\} \to \mathbb{R}\setminus\{0\}$, $(f \circ f)(x) = f(f(x))$

$$= \frac{1}{f(x)} = \frac{1}{1/x} = x.$$

<u>Remark.</u> $f \circ f$ is not equal to $I: \mathbb{R} \to \mathbb{R}$, $I(x) = x$ as they have different (co) domains.