Lecture 14: Quantifiers

Monday, October 24, 2016 9:22 AM

In the previous lecture we said how we can say $A \subseteq \mathbb{R}$ has a minimum.

Ex. Use quantifiers to say a nonempty subset A of $\mathbb{R}$ is bounded.

Solution. $\exists\, m, M \in \mathbb{R},\ \forall\, a \in A,\ \ m \le a \le M.$

a lower bound   an upper bound.

Ex. Prove or disprove that any bounded non-empty subset $A \subseteq \mathbb{R}$ has a minimum.

Solution. We disprove it by proving that $(0,1)$ is bound, but it does not have a minimum.

Boundedness. $(0,1)$ is bounded.

Proof. $\forall a \in (0,1)$, $0 \le a \le 1$ so in the above definition we can set $m=0$ and $M=1$. $\square$

No minimum $(0,1)$ has no minimum.

Proof. Suppose to the contrary that $(0,1)$ has a

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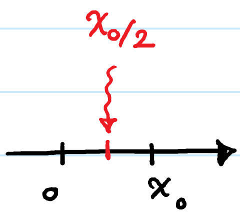
(*) $\exists x_0 \in (0,1),\ \forall y \in (0,1),\quad x_0 \le y.$

Since $x_0 \in (0,1)$, $0 < x_0$.



So $0 < \frac{x_0}{2}$ and $\frac{x_0}{2} < \frac{x_0}{2} + \frac{x_0}{2} = x_0 < 1$

So $\frac{x_0}{2} \in (0,1) \wedge x_0 \not\le \frac{x_0}{2}$

which contradics (*). ■

To get a better understanding of quantifiers, let's play a little bit.

<u>Ex.</u> I play against you. At each turn we have to say one of the numbers 1, 2, 3, 4, 5. A player is the winner if the mentioned numbers add up to 30 after his turn.

(Alternatively: There is a heap of coins which contains 30 coins. At each turn a player should pick either 1, 2, 3, 4, or 5 coins. A player wins if s/he picks the last coin.)

Determine if the first player can win or not.

<u>Solution</u>. This is a losing game (for the 1st player). To show this

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We have to show for any move of the 1^{st} player there is a suitable move for the 2^{nd} player.

Suppose $G(n)$ is a game with the same rules which has n coins at the initial stage, and let's call the players A and B. And suppose A plays first and takes $\underline{k}$ coins.

So here are the old and the new situations:

1^{st} player	2^{nd} player	game	
A	B	$G(n)$	old
B	A	$G(n-k)$	new

So $G(n)$ is a losing game for A if any move of A changes the game to a winning game for B. In this example it means

$G(n)$ is a <u>losing game</u> (for the 1^{st} player) exactly when $\underline{G(n-1)}$, $\underline{G(n-2)}$, $\underline{G(n-3)}$, $\underline{G(n-4)}$, and $\underline{G(n-5)}$ are <u>winning</u> games (for the 1^{st} player).

Using this rule we can easily figure out all the losing and winning

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For $1 \le n \le 5$, the 1^{st} player takes all the coins in her first move. So $G(1), G(2), G(3), G(4), G(5)$ are winning games. Using the above mentioned rule we get that $G(6)$ is a losing game.

Let's look at the "contrapositive" of the above mentioned rule

$G(n)$ is a winning game exactly when

$G(n-1)$ is losing <u>or</u> $G(n-2)$ is losing <u>or</u> $\cdots$ <u>or</u> $G(n-5)$ is losing.

Since $G(6)$ is losing, using the "blue rule" we get

$G(7), G(8), G(9), G(10), G(11)$ are winning games.

Hence again using the first rule $G(12)$ is losing. So we get the following pattern that you can prove using strong induction:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$G(n)$	P	P	P	P	P	N	P	P	P	P	P	N	P	P	P	P	P	N	P

where P means it is a winning game for the 1^{st} player and N means it is a losing game for the 1^{st} player.

Basically <u>$G(n)$ is N if and only if $6 \mid n$.</u> In particular, $G(30)$ is a losing game for the 1^{st} player. $\blacksquare$

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Going through the above argument you can also learn how you should actually play to win. Say we are playing G(11) and you are the 1^{st} player. By the above argument, we know you should be able to win this game. This means $\exists$ a suitable move for you. It does <u>NOT</u> mean if you play $\forall$ move you will end up winning.

What is your <u>suitable</u> move? You should make a move and change the game to a losing game for your opponent: change 11 to a multiple of 6. So take 5 coins.

Now no matter what your opponent does, he is going to lose.

Let's consider G(12), and suppose you are the 2^{nd} player. So by the above discussion again you are supposed to be able to win this game. It means:

$\forall$ move of your opponent $\exists$ a suitable move for you.

In this case we can give a concrete relation between your opponents move and your response:

If your opponent takes k coins, then you take $6-k$ coins.

So after your move there will be 6 remaining coins which is a losing game for your opponent.

As it should be, you can see that your move depends on your opponents move. This is always the case when you are dealing with a proposition of the following form:

$\forall x \in X, \exists y \in Y,$ certain property of x and y holds.

Let's expand on how we studied $G(n)$ and go over the basics of (combinatorial) game theory.

Definition. By a game G we mean

1. a game between two players.
2. Players play in turns. (After A plays, it is B's turn, and vice versa.)
3. It is a finite game, i.e. there is an integer $\ell(G)$ such that we know after $\ell(G)$-turns the game is over.
4. A player who makes the last possible move is the winner. (A player who cannot make a (legal) move is the loser.)

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<u>Definition</u>. A game G is called a <u>winning game</u>, denoted by $\mathcal{P}$, if it is a winning game for the 1^{st} player.

. A game G is called a <u>losing game</u>, denoted by N, if it is a losing game for the 2^{nd} player.

By similar arguments an in the analysis of G(n) we have:

<u>A game is $\mathcal{P}$</u> exactly when $\exists$ a suitable move for the 1^{st} player

which is equivalent to saying $\exists$ a move for the 1^{st} player which changes the game to N.

This can be said as

$$\exists \text{ move} \in \{\text{possible moves for } A\}, \forall \text{ move} \in \{\text{possible moves for } B\}, \text{ A can win.}$$

Similarly

A game is N exactly when

$$\forall \text{ move} \in \{\text{possible moves for } A\}, \exists \text{ move} \in \{\text{possible moves for } B\}, \text{ A cannot win.}$$

<u>Definition</u> Let G_1, G_2 be two games. Then $G_1 \oplus G_2$ is a game where at each turn a player should play in either G_1

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or G_2. S/he cannot play at both G_1 and G_2 in one move.

Ex. Determine if $G(1) \oplus G(1)$ is P or N.

Solution. It is a losing game (N). After the 1st player's move, we end up with 1 coin, and so the 2nd player finishes the game. ∎

Ex. For any game G, $G \oplus G$ is a losing game.

"proof" For any move of the 1st player, we need to find a suitable move.

The suitable move of the 2nd player is to "mirror" the 1st player's move. (This proof is not written in a formal way. Can you make it precise?) □

The main reason that we discussed basics of game theory was to see that ① we naturally think about repeated quantifiers when we are playing a stratagic game. ② I believe interpreting propositions of the form $\exists x \in X, \forall y \in Y, \cdots$ as a winning game and

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$\forall x \in X, \exists y \in Y, \cdots$ as a losing game helps us to to find their negations, and remember that in the first case x should work for all the choices of y. And in the second case for any given x we should find a suitable y to work.