Lecture 13: Properties of set operations

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At the end of the previous lecture we were proving

<u>Lemma</u>. Suppose X is a set. Then for any $A, B \subseteq X$ we have

$$A \triangle B = (A\cup B)\setminus(A\cap B).$$

[<u>Recall</u>. $A\triangle B = (A\setminus B)\cup(B\setminus A)$.].

<u>Proof</u>.

$x\in A \vee x\in B$ | $x\in A \wedge (\neg x\in B)$ | $x\in A\setminus B \vee x\in B\setminus A$

$x\in A$	$x\in B$	$x\in A\cup B$	$x\in A\cap B$	$x\in (A\cup B)\setminus(A\cap B)$	$x\in A\setminus B$	$x\in B\setminus A$	$x\in A\triangle B$
T	T	T	T	F	F	F	F
T	F	T	F	T	T	F	T
F	T	T	F	T	F	T	T
F	F	F	F	F	F	F	F

$x\in A \wedge x\in B$ | $x\in B \wedge (\neg x\in A)$

$x\in A\cup B \wedge (\neg\, x\in A\cap B)$

Hence, for any $x\in X$,

$$x\in (A\cup B)\setminus(A\cap B) \iff x\in A\triangle B,$$

which implies $(A\cup B)\setminus(A\cap B) = A\triangle B$. ∎

As in the case of propositional forms, there are extremely useful set equalities. Before I write some of them, let me introduce complement of a subset A of X. It is denoted by A^c:

$$A^c = \{x\in X \mid x\notin A\}.$$ So we have $A^c = X\setminus A$.

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<u>Theorem</u>. Suppose X is a set. For any $A, B, C \subseteq X$, we have

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(1)′ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(2) $(A \cup B)^c = A^c \cap B^c$

(2)′ $(A \cap B)^c = A^c \cup B^c$

(3) $A \setminus B = A \cap B^c$

(4) $A \cap B \subseteq A \subseteq A \cup B$ and $A \cap B \subseteq B \subseteq A \cup B$.

(5) $A \subseteq B \overset{(a)}{\iff} A \cap B = A$

$\overset{(b)}{\iff} A \cup B = A$

$\overset{(c)}{\iff} A \setminus B = \emptyset$

(6) $A \cap B = \emptyset \iff A \subseteq B^c$.

<u>Proof</u>. For any $x \in X$,

(1) $x \in A \cup (B \cap C) \iff x \in A \vee x \in B \cap C$

$\iff x \in A \vee (x \in B \wedge x \in C)$

$\iff (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$

$\iff x \in A \cup B \wedge x \in A \cup C$

$\iff x \in (A \cup B) \cap (A \cup C)$.

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Hence $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

(1)' is similar to (1).

(2) $x\in(A\cup B)^c \Leftrightarrow x\notin A\cup B \Leftrightarrow \neg(x\in A\cup B)$

$\Leftrightarrow \neg(x\in A \vee x\in B)$

Remember that $x\in X$

$\Leftrightarrow \neg(x\in A) \wedge \neg(x\in B)$

$\Leftrightarrow x\in A^c \wedge x\in B^c$

$\Leftrightarrow x\in A^c\cap B^c$.

So $(A\cup B)^c = A^c\cap B^c$.

(2)' is similar to (2).

Remember $x\in X$

(3) $x\in A\setminus B \Leftrightarrow x\in A \wedge x\notin B \Leftrightarrow x\in A \wedge x\in B^c$

$\Leftrightarrow x\in A\cap B^c$.

Therefore $A\setminus B = A\cap B^c$.

(4) $x\in A\cap B \Rightarrow x\in A \wedge x\in B$

$\Rightarrow x\in A$. Thus $A\cap B\subseteq A$.

$x\in A \Rightarrow x\in A \vee x\in B$

$\Rightarrow x\in A\cup B$. So $A\subseteq A\cup B$.

(5)((a) $\Rightarrow$) We have to show $A\cap B\subseteq A$ and $A\subseteq A\cap B$.

The former is proved in (4).

$x\in A \Rightarrow x\in B$ since $A\subseteq B$. So $x\in A \Rightarrow (x\in A \wedge x\in B)$

$\Rightarrow x\in A\cap B$.

Hence $A\subseteq A\cap B$.

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($\overset{(a)}{\Leftarrow}$) By (4) we have $A\cap B \subseteq B$. By assumption $A = A\cap B$. Hence $A \subseteq B$.

($\overset{(b)}{\Rightarrow}$) We have to prove $B \subseteq A\cup B$ and $A\cup B \subseteq B$.

The former is proved in (4).

$x \in A\cup B \Rightarrow x\in A \vee x \in B$

<u>Case 1.</u> $x \in A \Rightarrow x \in B$ since $A \subseteq B$.

<u>Case 2.</u> $x \in B \Rightarrow x \in B$.

So in either case we conclude $x \in B$. Therefore $x \in A\cup B \Rightarrow x \in B$. So $A\cup B \subseteq B$.

($\overset{(b)}{\Leftarrow}$) By (4) we have $A \subseteq A\cup B$. By the assumption, $A\cup B = B$. Hence $A \subseteq B$.

($\overset{(c)}{\Rightarrow}$) We have to prove $A \subseteq B \Rightarrow A\setminus B = \emptyset$.

Suppose to the contrary for some subsets A and B we have $A \subseteq B \wedge A\setminus B \neq \emptyset$.

$A\setminus B \neq \emptyset \Rightarrow$ there exists $x_0 \in A\setminus B$

$\Rightarrow x_0 \in A \wedge x_0 \notin B$

$\Rightarrow x_0 \in B \wedge x_0 \notin B$ as $A \subseteq B$.

This is a contradiction.

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($\overset{(\Leftarrow)}{\Longleftarrow}$) We have to show $A \setminus B = \emptyset \Rightarrow A \subseteq B$.

Suppose to the contrary there are subsets A and B such that

$A \setminus B = \emptyset \wedge A \not\subseteq B$.

So $\neg($ for <u>any</u> x, $x \in A \Rightarrow x \in B)$, which implies

for some x_0, $x_0 \in A \wedge x_0 \notin B$.

$\Rightarrow x_0 \in A \setminus B \Rightarrow A \setminus B \neq \emptyset$ which contradicts

our assumption.

(6) Using part (3) and the fact that $(B^c)^c = B$, we have

$A \cap B = A \setminus B^c$. By part (5c) we have

$A \subseteq B^c \Leftrightarrow A \setminus B^c = \emptyset$.

Therefore $A \subseteq B^c \Leftrightarrow A \cap B = \emptyset$. ■

[In class we went through only parts (4), (5), and (6), but I expect you to go over the rest of proof.]

<u>Quantifiers</u>

Most of mathematical results involve <u>quantifiers</u>. They help us understand in what capacity should we look at a variable as a member of a set.

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We might say for any x in X, ... or for <u>all</u> x in X, ...

Taking the capital letter A from the word <u>all</u> and flipping it we get the mathematical symbol $\forall$ for this quanifier. This is called the universal quantifier.

Another type of quantifier is for some x in X, ... or alternatively there exists x in X, ...

Taking the capital letter E from the word <u>exists</u> and flipping it we get the mathematical symbol $\exists$ for this quantifier. This is called the existential quantifier.

<u>Ex.</u> To say 2 is prime is equivalent to

$$\forall m,n\in\mathbb{Z},\ 2\mid mn \Rightarrow (2\mid m \vee 2\mid n).$$

<u>Ex.</u> Suppose $A\subseteq\mathbb{R}$. Use mathematical language to say A has a minimum.

<u>Solution</u> $\exists x\in A, \forall y\in A, x\leq y.$

x is supposed to be the minimum of A; it means for any element y of A, x should be less than or equal to y. ■