Lecture 12: Subsets, cardinality of finite sets

Thursday, October 20, 2016 2:53 PM

In the previous lecture we said what it means to say A is a subset of B and it is denoted by $A \subseteq B$.

It means "$x \in A \Rightarrow x \in B$".

So for two sets A and B, we have

$$A = B \Leftrightarrow (A \subseteq B \wedge B \subseteq A).$$

Ex.. For any set A, we have $A \subseteq A$ and $\emptyset \subseteq A$.

- $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.
- Using an axiom of set theory we have that, for any set A, $A \notin A$. [Recall. $\neg(x \in X)$ is denoted by $x \notin X$.]
- $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

"Definition" For a finite set X, the number of elements of X is called the cardinality of X. And it is denoted by $|X|$.

Ex.. $|\{1,1\}| = 1$. • $|\{1, 2, \{1,2\}\}| = 3$.

- $|\emptyset| = 0$.
- $|\{\{1\}, \{1,1\}\}| = 1$. In this example we are using the fact that $\{1\} = \{1,1\}$, and so $\{\{1\},\{1,1\}\} = \{\{1\}\}$.

Lecture 12: Power set

Thursday, October 20, 2016 3:28 PM

Ex.. $|\{1,2,\mathbb{R}\}| = 3$. Elements of this set are 1, 2, and $\mathbb{R}$.

- $|\{1,2,\mathbb{R},\mathbb{C}\}| = 4$.
- $|\{\{-1,1\},\{x\in\mathbb{R} \mid x^2=1\}\}| = 1$.

  Here we are using the fact that $\{x\in\mathbb{R} \mid x^2=1\} = \{-1,1\}$.

Definition. For a set X, the set of subsets of X is called its power set, and it is denoted by $\mathcal{P}(X)$. So

$$\mathcal{P}(X) = \{A \mid A\subseteq X\}.$$

Ex.. $\mathcal{P}(\emptyset) = \{\emptyset\}$. So $|\mathcal{P}(\emptyset)| = 1$.

- $\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$. So $|\mathcal{P}(\{1\})| = 2$.
- For any set X, $\{\emptyset, X\} \subseteq \mathcal{P}(X)$.
- For any sets A, X, $A\subseteq X \Leftrightarrow A\in \mathcal{P}(X)$.
- $\mathcal{P}(\{a,b\})$

In order to list all the subsets of $\{a,b\}$, we can think about a subset as a "club" and we have to decide who will be its member and who will not. We decide if $\underline{\underline{a}}$ should be in this "club", and if $\underline{\underline{b}}$ should be in this "club" or not.

Lecture 12: Power set

Thursday, October 20, 2016 3:44 PM

We use a truth table to list all the possible "decisions".

$a \in A$	$b \in A$	
T	T	$\rightsquigarrow \{a,b\}$
T	F	$\rightsquigarrow \{a\}$
F	T	$\rightsquigarrow \{b\}$
F	F	$\rightsquigarrow \emptyset$

So $\mathcal{P}(\{a,b\})$

$= \{\emptyset, \{b\}, \{a\}, \{a,b\}\}$.

Ex. $\mathcal{P}(\{1,2,\{1,2\}\}) = ?$

$1 \in A$	$2 \in A$	$\{1,2\} \in A$	
T	T	T	$\rightsquigarrow \{1,2,\{1,2\}\}$
T	T	F	$\rightsquigarrow \{1,2\}$
T	F	T	$\rightsquigarrow \{1,\{1,2\}\}$
T	F	F	$\rightsquigarrow \{1\}$
F	T	T	$\rightsquigarrow \{2,\{1,2\}\}$
F	T	F	$\rightsquigarrow \{2\}$
F	F	T	$\rightsquigarrow \{\{1,2\}\}$
F	F	F	$\rightsquigarrow \emptyset$

So $\mathcal{P}(\{1,2,\{1,2\}\}) = \{\emptyset, \{\{1,2\}\}, \{2\}, \{2,\{1,2\}\},$

$\{1\}, \{1,\{1,2\}\}, \{1,2\}, \{1,2,\{1,2\}\}\}$.

In particular $|\mathcal{P}(\{1,2,\{1,2\}\})| = 8$.

The same line of argument implies that to list elements of

$\mathcal{P}(\{a_1,a_2,\dots,a_n\})$ we have write a "truth table" for deciding

2 choices $\times$ 2 choices $\times \cdots \times$ 2 choices $= 2^n$ possibilities.

$a_1 \in A$, $a_2 \in A, \dots, a_n \in A$. Hence $|\mathcal{P}(X)| = 2^{|X|}$.

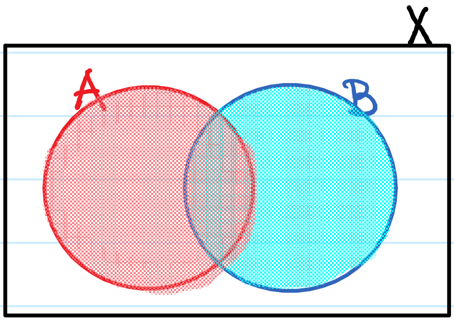
Lecture 12: Power set and set operations

Thursday, October 20, 2016 4:01 PM

We will make a formal proof later, but its idea is basically what is presented here.

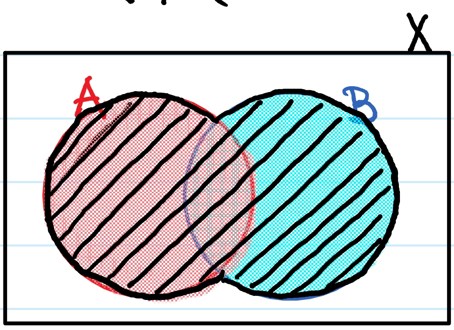
In order to have a better feeling about sets, sometimes we use <u>Venn diagrams</u>:

Suppose X is a set and $A, B \subseteq X$. The associated Venn diagram looks like:



Thinking about A and B as two "student clubs", we can form a new "club" by merging them. It is called the union of A and B. It is denoted by $A \cup B$. So, for any $x \in X$,

$$x \in A \cup B \iff (x \in A \vee x \in B).$$

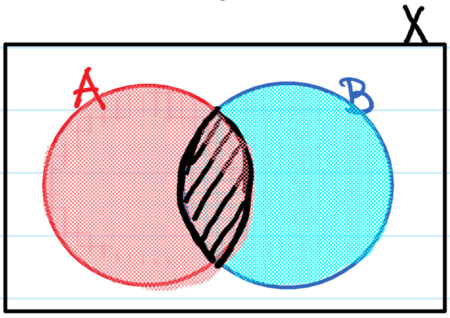


Lecture 12: Set operations

Thursday, October 20, 2016 4:18 PM

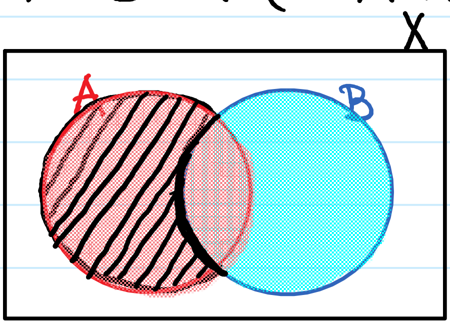
Again thinking about A and B as two "student clubs", we can form a new club out of students who are in both of the clubs. It is called the intersection of A and B and it is denoted by $A \cap B$. So for any $x \in X$

$$x \in A \cap B \Leftrightarrow (x \in A \wedge x \in B).$$



Thinking about A as a "student club" and B as "bad students"(!), we can form a new club out of members who are NOT bad. It is called the set difference of A and B, and it is either denoted by $A \setminus B$ or $A - B$. I will be using $A \setminus B$. So for any $x \in X$

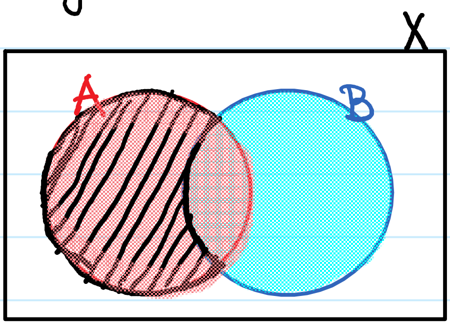
$$x \in A \setminus B \Leftrightarrow (x \in A \wedge x \notin B).$$



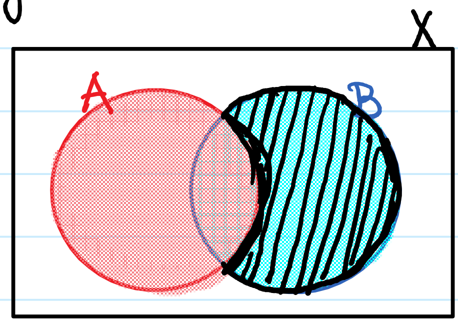
Lecture 12: Set operation

Thursday, October 20, 2016 4:37 PM

As you can $A\setminus B$ has nothing to do with $B\setminus A$.

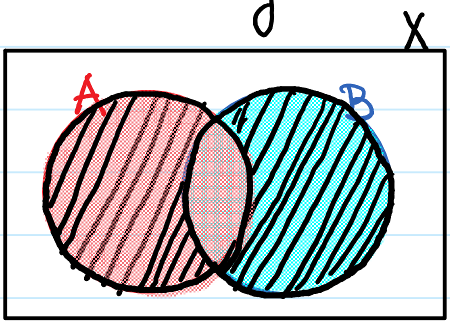


$A\setminus B$



$B\setminus A$

An important set operation is the symmetric difference of A and B. It is denoted by $A\triangle B$. Its Venn diagram is as follows



It can be defined as $A\triangle B = (A\setminus B)\cup(B\setminus A)$.

In 100 or 103, you will learn that $(\mathcal{P}(X), \triangle)$ form a group.

In the next lecture we use truth-table to show

<u>Lemma</u>. For any $A,B\subseteq X$, we have

$$A\triangle B = (A\cup B)\setminus(A\cap B)$$

we complete a table to find out the last two columns are identical.

$x\in A$	$x\in B$	$x\in A\triangle B$	$x\in(A\cup B)\setminus(A\cap B)$
T	T		
T	F		
F	T		
F	F		