Lecture 9: Binet; Eigenvalue

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Theorem (Binet) For any non-negative integer n, we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right).$$

Before we prove this theorem using strong induction, let's see a slightly better approach which relies on a little bit of knowledge of linear algebra.

In the previous lecture we proved that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

So, if we manage to compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ in another way, we get a formula for F_n.

If we diagonalize $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then we can compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$.

In order to diagonalize a matrix, one has to compute its eigenvalues and eigenvectors. So we have to find roots of

$$\det\left(xI - \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 \Rightarrow \det\begin{bmatrix} x-1 & -1 \\ -1 & x \end{bmatrix} = 0$$

$\Rightarrow$ $x^2 - x - 1 = 0$. And its roots are

$$\beta = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \alpha = \frac{1-\sqrt{5}}{2}.$$

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Hence for some matrix S we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = S \begin{bmatrix} \beta & \\ & \alpha \end{bmatrix} S^{-1}.$$

Therefore

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(S \begin{bmatrix} \beta & \\ & \alpha \end{bmatrix} \cancel{S^{-1}}\right)\left(\cancel{S} \begin{bmatrix} \beta & \\ & \alpha \end{bmatrix} \cancel{S^{-1}}\right) \cdots \left(\cancel{S} \begin{bmatrix} \beta & \\ & \alpha \end{bmatrix} S^{-1}\right)$$

$$= S \begin{bmatrix} \beta & \\ & \alpha \end{bmatrix}^n S^{-1} = S \begin{bmatrix} \beta^n & \\ & \alpha^n \end{bmatrix} S^{-1}.$$

Thus $S \begin{bmatrix} \beta^n & \\ & \alpha^n \end{bmatrix} S^{-1} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$.

Now either one can find S by computing eigenvectors of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

or one can observe that the above equality implies

$F_n = c_1 \beta^n + c_2 \alpha^n$ for some constants c_1 and c_2. And

then find c_1 and c_2 using $\begin{cases} F_0 = 0 = c_1 + c_2 \\ F_1 = 1 = c_1 \beta + c_2 \alpha. \end{cases}$

Now let's try to prove this formula using <u>induction</u>. The base case is clear. For the induction step, we assume

$F_k = \frac{1}{\sqrt{5}}(\beta^k - \alpha^k)$ and we have to prove $F_{k+1} = \frac{1}{\sqrt{5}}(\beta^{k+1} - \alpha^{k+1})$

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But F_{k+1} depends on both F_k and F_{k-1}. So the induction hypethesis which gives us information about only one step back does NOT help us.

Strong induction. In order to prove

for any positive integer n, $P(n)$ holds.

It is enough to prove

① (Base case) $P(1)$ holds.

② (Strong induction step) For a given positive integer k, assume $P(i)$ holds for $1 \leq i \leq k$. Then we have to prove $P(k+1)$ holds.

(informally: $\left(P(1) \wedge P(2) \wedge \cdots \wedge P(k)\right) \Rightarrow P(k+1)$.)

In the Binet formula, we have prove certain property holds for all non-negative integers instead of positive integers. This only effects the initial value. The rest stays the same. Here is its formulation:

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Strong induction, initial value n_0 In order to prove for any integer $n \geq n_0$, $P(n)$ holds.

It is enough to prove

① (Base case) $P(n_0)$ holds.

② (Strong induction step) For a given integer $k \geq n_0$ assume $P(i)$ holds for $n_0 \leq i \leq k$. Then we have to prove $P(k+1)$ holds.

(informally. $\left(P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k)\right) \Rightarrow P(k+1)$.)

Proof of Binet using strong induction.

We use strong induction on n.

Base of strong induction $n=0$ as we are proving the claimed equality for any non-negative integer. So we have to show

$F_0 = \frac{1}{\sqrt{5}}(\beta^0 - \alpha^0)$. The LHS $=0$ and the RHS $= \frac{1}{\sqrt{5}}(1-1) = 0$

The strong induction step. For a given non-negative integer k, we assume $F_i = \frac{1}{\sqrt{5}}(\beta^i - \alpha^i)$ for any $0 \leq i \leq k$. We have to show $F_{k+1} = \frac{1}{\sqrt{5}}(\beta^{k+1} - \alpha^{k+1})$.

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In order to find F_{k+1} we notice that, if $\underline{\underline{k>0}}$, then

$F_{k+1}=F_k+F_{k-1}$. So we consider two cases:

Case 1. $k=0$. Then $F_{k+1}=F_1=1$ and

$$\frac{1}{\sqrt{5}}\left(\beta^{k+1}-\alpha^{k+1}\right)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)\right]=1.$$ So

we get $F_{k+1}=\frac{1}{\sqrt{5}}\left(\beta^{k+1}-\alpha^{k+1}\right)$.

Case 2. $k>0$. Then

$$F_{k+1}=F_k+F_{k-1}=\frac{1}{\sqrt{5}}\left(\beta^k-\alpha^k\right)+\frac{1}{\sqrt{5}}\left(\beta^{k-1}-\alpha^{k-1}\right)$$

by the strong induction hypothesis

$$=\frac{1}{\sqrt{5}}\left(\beta^k+\beta^{k-1}-\alpha^k-\alpha^{k-1}\right)$$

$$=\frac{1}{\sqrt{5}}\left[\beta^{k-1}(\beta+1)-\alpha^{k-1}(\alpha+1)\right]$$

Since α and β are roots of $x^2-x-1=0$, we have

$\alpha^2=\alpha+1$ and $\beta^2=\beta+1$. Hence

$$F_{k+1}=\frac{1}{\sqrt{5}}\left[\beta^{k-1}\cdot\beta^2-\alpha^{k-1}\cdot\alpha^2\right]=\frac{1}{\sqrt{5}}\left(\beta^{k+1}-\alpha^{k+1}\right).$$ ∎