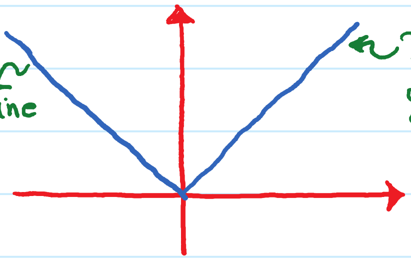
Lecture 6: Basic properties of absolute value

Friday, October 7, 2016 12:17 AM

TAs informed me that some of you had difficulty on formulating a formal definition for $|x|$. Let me recall from calculus that the graph of $y=|x|$ looks like



So $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

One of the important properties of absolute value is the following:

Lemma For any real numbers x and y,

$$|xy| = |x||y|.$$

Proof. If $x=0$, then $xy=0 \Rightarrow |xy|=0$

and $|x|=0 \Rightarrow |x||y|=0$.

So $|xy|=|x||y|$.

If $y=0$, then by symmetry that $|xy|=|x||y|$.

Changing x to y and y to x does NOT change the hypoth. so whatever proved for x can be proved for y.

Using this line of argument one has to be extra careful. Most of errors in math articles occure in this type of arguments.

Lecture 6: multiplicativity of absolute value

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x	y	xy	$\lvert x\rvert$	$\lvert y\rvert$	$\lvert xy\rvert$	$\lvert x\rvert\lvert y\rvert$
$+$	$+$	$+$	x	y	xy	xy
$+$	$-$	$-$	x	$-y$	$-xy$	$-xy$
$-$	$+$	$-$	$-x$	y	$-xy$	$-xy$
$-$	$-$	$+$	$-x$	$-y$	xy	xy

So in the rest of proof we can and will assume $x \neq 0$ and $y \neq 0$. So they are either positive or negative.

The above table shows us all the possibilities of the values of $|x||y|$ and $|xy|$ depending on the signs of x and y.

Looking at the highlighted columns, we see that

$$|xy| = |x||y|$$

in all the remaining cases. ■

<u>Corollary 1</u>. $|x|^2 = |x^2| = x^2$ for any real number x.

<u>Proof</u>. Let $y = x$ in the above lemma to get $|x|^2 = |x^2|$.

For any real number x, $x^2 \geq 0$ (why?). So $|x^2| = x^2$. ■

<u>Corollary 2</u>. For any real number x, $\sqrt{x^2} = |x|$.

<u>Proof</u>. By definition, $z = \sqrt{y} \Leftrightarrow (z \geq 0 \wedge z^2 = y)$.

We notice that $|x| \geq 0$ and $|x|^2 = x^2$ by Corollary 1. ■

Lecture 6: A simple inequality.

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<u>Lemma</u>. For any real numbers x, y,

$$x^2 \le y^2 \iff |x| \le |y|.$$

<u>Proof</u>. ($\Rightarrow$) $x^2 \le y^2 \Rightarrow |x|^2 \le |y|^2$ (by Corollary 1)

$$\Rightarrow 0 \le |y|^2 - |x|^2 = (|y|-|x|)(|y|+|x|).$$

Since $|x| \ge 0$ and $|y| \ge 0$, we have that either $x=y=0$ or $|x|+|y|>0$.

<u>Case 1</u>. $x=y=0$.

In this case $|x|=0$ and $|y|=0$. So $|x| \le |y|$.

<u>Case 2</u>. $|x|+|y|>0$.

Since the product of a <u>positive number</u> $|x|+|y|$ by $|y|-|x|$ is <u>non-negative</u>, $|y|-|x|$ should be non-negative. So $|y|-|x| \ge 0$ which implies

$$|y| \ge |x|.$$

($\Leftarrow$) For this direction, we use a backward argument:

$$x^2 \le y^2 \Leftarrow |x|^2 \le |y|^2 \Leftarrow 0 \le (|y|-|x|)(|y|+|x|) \Leftarrow \begin{cases} |x| \le |y| \\ |x| \ge 0 \\ |y| \ge 0. \end{cases} \blacksquare$$

Lecture 6: Unofficial introduction to induction

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Q. What is $1+3+5+\dots+(2n-1)$ where n is a positive integer?

As always when you are faced by a new problem, start by some examples: in this case small numbers.

$n=1 \rightsquigarrow 1$.

$n=2 \rightsquigarrow 1+3=4$.

$n=3 \rightsquigarrow 1+3+5=9$.

$n=4 \rightsquigarrow 1+3+5+7=16$

At this stage you might be able to guess a formula: Yes.

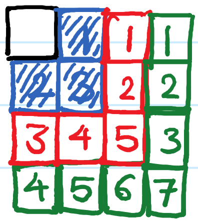
"Conjecture": $1+3+\dots+(2n-1)=n^2$.

"n squared". So let's try to visualize it by creating a square

$n=1$

$n=2$

3 extra



$n=3$

5 extra

$n=4$

7 extra

Maybe we can continue like this! Let's see how many little squares are needed to go from an $n\times n$ square to an $(n+1)\times(n+1)$ square.

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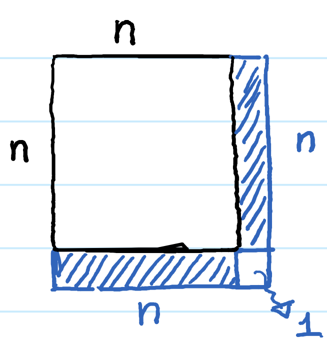

So $\underline{2n+1}$ small squares are added which is exactly the next odd number after $2n-1$.

$\underline{Q}$. How can we make sense of the following number? What is it?

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}\,.$$

Whenever you see $\cdots$ (and so on), it means there is a pattern and we are continuing accordingly. Let's try to understand this pattern:

$a_1=\sqrt{2}$.

$a_2=\sqrt{2+\sqrt{2}}$

$a_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$

$a_4=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$

Looking at these we should be able to guess the pattern:

$$a_{n+1}=\sqrt{2+a_n}\,.$$

Let $f(x)=\sqrt{2+x}$.

Then $a_{n+1}=f(a_n)$. So each time we are applying the function f to get the next number. Let's try to visualize

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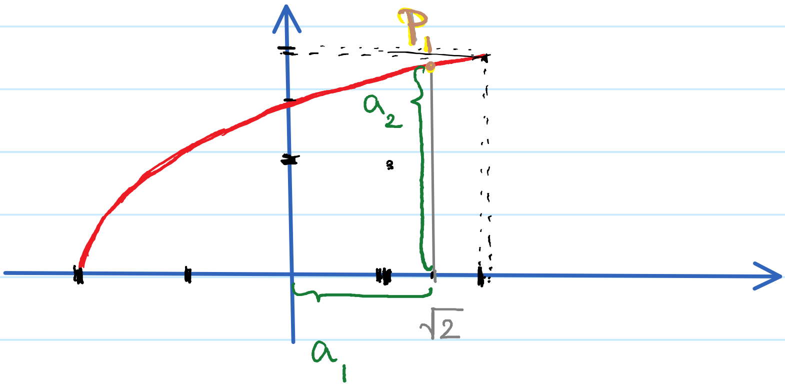
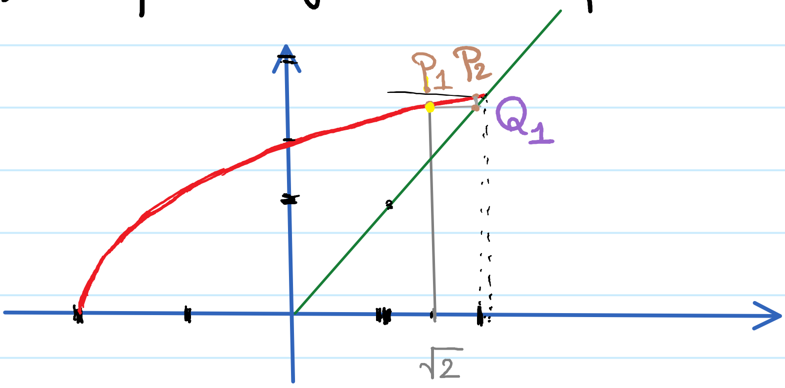
this process. So we start with graph $y=f(x)$ of f.



We notice that $P_1=(a_1, f(a_1))=(a_1,a_2)$. Now in order to find $\underline{a_3}$ we need to find $\underline{a_2}$ on the x-axis, (instead of y-axis). Graph of $y=x$ can help us on this.



Drawing a segment parallel to the x-axis from $P_1=(a_1,a_2)$ till hitting the line $y=x$, we end up getting to the point $Q_1=(a_2,a_2)$. Now going parallel to the y-axis from $Q_1=(a_2,a_2)$ till hitting the graph $y=f(x)$, we end up getting to the point $P_2=(a_2,f(a_2))=(a_2,a_3)$. And we can continue like

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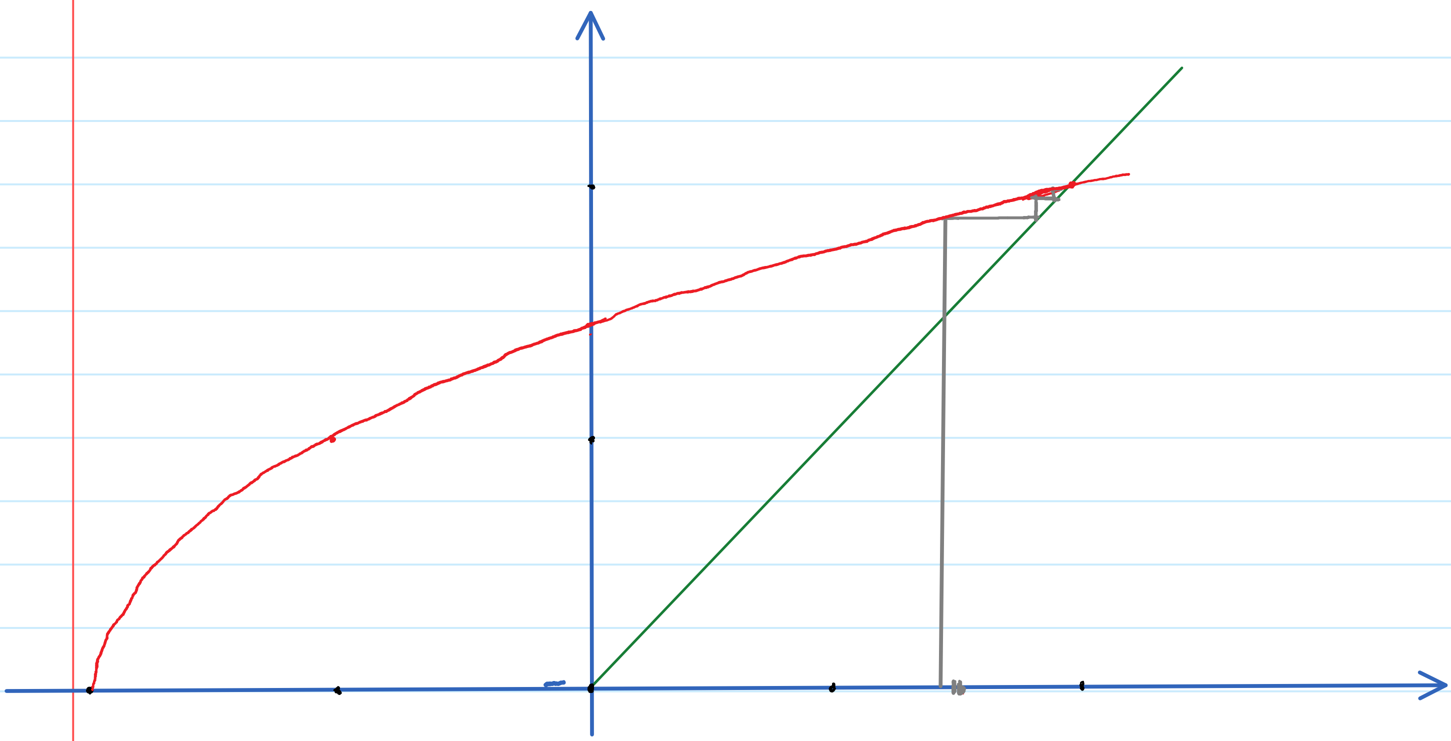

From this picture we can "conjecture" that the points (a_n, a_{n+1}) are getting closer and closer to the point of intersection of $y=\sqrt{2+x}$ and $y=x$.

What is this point?

$$\sqrt{2+x} = x \Rightarrow 2+x = x^2$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \text{ or } x = -1.$$

Since $x \geq 0$, we get that $x = 2$.

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So we conjecture that $a_n \to 2$ as $n \to \infty$.

From this picture we conjecture that for any positive integer n

① $a_n < a_{n+1}$

② $a_n < 2$

In the next lecture we will prove these claims.