Lecture 5: (Continuation) description of odd numbers

Monday, October 3, 2016 9:22 AM

We were in the middle of a proof of:

<u>Lemma</u>. For any integer n, n is odd if and only if, for some integer k, we have $n = 2k+1$.

<u>Proof</u>. ($\Longrightarrow$) Supposed m is the largest even number such that $m \le n$.

Then there is an integer k such that $m = 2k$.

Let $r = n - m$. Then $0 \le r$.

<u>Claim 1</u>. $r \ne 0$.

<u>Proof of claim 1</u>. Suppose to the contrary that $r = 0$. Then $n = m$ which implies n is even. This contradicts our assumption. ∎

<u>Claim 2.</u> $r \le 1$.

<u>Proof of Claim 2.</u> Suppose to the contrary that $r > 1$. Since r is an integer, we get $r \ge 2$. Hence $m+2 \le m+r = n$.

So $2k+2 = 2(k+1) \le n$ is an even number <u>larger than m</u> which is at most n. This contradicts the way we chose m. ∎

Since r is an integer and $0 < r \le 1$, we have $r = 1$.
Hence $n = 2k+1$.

Lecture 5: What is prime?

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($\Leftarrow$) Suppose $n = 2k+1$ for some integer k, we want to prove n is odd, i.e. n is NOT even. Suppose to the contrary n is even. Then for some integer k' we have

$$2k+1 = 2k'.$$

Hence $1 = 2k' - 2k = 2(k'-k)$. Since $k'-k$ is an integer, we have $2 \mid 1$. Therefore we should have $|2| \leq |1|$ which is a contradiction. ■

<u>Definition</u>. An integer $p > 1$ is called <u>prime</u> if the following holds: for any integers a, b,

$$p \mid ab \Rightarrow (p \mid a \vee p \mid b).$$

<u>Remark</u>. You have seen another definition for prime. In the algebra series you will see that the definition that you have seen before is going to be called <u>irreducible</u>:

$$p = ab \Longrightarrow (a = \pm p \vee b = \pm p).$$

For integers, we will see that these are equivalent, but not for any "system of numbers" (called ring).

Lecture 5: 2 is prime.

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<u>Theorem</u> 2 is prime.

<u>Proof.</u> Suppose to the contrary that 2 is NOT prime. So <u>for some</u> integers a and b,

$$2 \mid ab \ \wedge \ 2 \nmid a \ \wedge \ 2 \nmid b.$$

$$\left.\begin{array}{l} 2 \nmid a \implies \text{for some integer } k, \quad a = 2k+1 \\ 2 \nmid b \implies \text{for some integer } l, \quad b = 2l+1 \end{array}\right\} \circledast$$

By $\circledast$, $ab = (2k+1)(2l+1) = 4kl + 2k + 2l + 1$

$$= 2\underbrace{(2kl + k + l)}_{\text{integer}} + 1$$

$\implies$ ab is of the form $2k'+1$ for some integer k'.

$\implies$ ab is odd which contradicts $2 \mid ab$. ■

<u>Corollary.</u> ab is odd if and only if a and b are odd.

<u>Corollary.</u> ab is even if and only if either a or b is even.

Lecture 5: Odd, even; inequality

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<u>Corollary</u>. For any integer n, n is odd if and only if

$n+1$ is even.

<u>Proof</u>. ($\Rightarrow$) n is odd $\Rightarrow$ $n = 2k+1$ for some integer k

$\Rightarrow n+1 = 2k+2 = 2\underbrace{(k+1)}_{\text{integer}} \Rightarrow 2 \mid n+1$

$\Rightarrow n+1$ is even.

($\Leftarrow$) $n+1$ is even $\Rightarrow 2 \mid n+1$

$\Rightarrow n+1 = 2k$ for some integer k

$\Rightarrow n = 2k-1 = 2\underbrace{(k-1)}_{\text{integer}}+1$

$\Rightarrow n$ is odd. $\blacksquare$

Inequalities are perfect examples of "backward" arguments:

<u>Theorem</u> For any real numbers x and y we have

$$x^2+y^2 \geq 2xy.$$

<u>Proof (Draft version; it should be rewritten from bottom to top.)</u>

$x^2+y^2 \geq 2xy \Leftarrow x^2+y^2-2xy \geq 0$

$\Leftarrow (x-y)^2 \geq 0.$

For any real number z, we have $z^2 \geq 0$. (why?) $\blacksquare$