Problem set 6

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1. Let A, B, C be three sets. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

2. (a) Prove or disprove:

$$\forall x \in \mathbb{R}, ((\forall \varepsilon > 0, \ |x| < \varepsilon) \Rightarrow x = 0).$$

(b) Prove or disprove:

$$\forall x \in \mathbb{R}, \forall \varepsilon > 0, \ (|x| < \varepsilon \Rightarrow x = 0).$$

(The first part of this problem you had in the previous problem set. The reason that it is repeated is for you to compare it with the 2^{nd} part.)

3. Let X be a non-empty set. The characteristic function of a subset A of X is defined as follows:

$$\mathbb{1}_A : X \to \{0, 1\}, \quad \mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

For instance, if $X = \{a, b\}$, then

x	$\mathbb{1}_{\emptyset}(x)$	$\mathbb{1}_{\{a\}}(x)$	$\mathbb{1}_{\{b\}}(x)$	$\mathbb{1}_{\{a,b\}}(x)$
a	0	1	0	1
b	0	0	1	1

For instance the 3^{rd} column says: $\mathbb{1}_{\{a\}}(a) = 1$, $\mathbb{1}_{\{a\}}(b) = 0$.

So essentially $\mathbb{1}_A(x)$ tells us whether or not $x \in A$. More precisely: $x \in A \Leftrightarrow \mathbb{1}_A(x) = 1$ and $x \notin A \Leftrightarrow \mathbb{1}_A(x) = 0$.

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(a) Prove that, for any $A, B \subseteq X$ and $x \in X$,

$$\mathbb{1}_{A\cap B}(x) = \mathbb{1}_A(x)\,\mathbb{1}_B(x).$$

(b) Prove that, for any $A \subseteq X$,

$$\mathbb{1}_A + \mathbb{1}_{A^c} = \mathbb{1}_X.$$

(c) Prove that for any $A, B \subseteq X$

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_A \cdot \mathbb{1}_B$$

(which means for any $x \in X$ we have

$$\mathbb{1}_{A\cup B}(x) = \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_A(x)\,\mathbb{1}_B(x).)$$

[Parts (a), (b), (c) can be proved using a truth-table type of argument in terms of $x \in A$, $x \in B$.]

(d) Prove that $\mathbb{1}_{A\setminus B} = \mathbb{1}_A - \mathbb{1}_A \cdot \mathbb{1}_B$

(Hint. Use $A\setminus B = A \cap B^c$ and previous parts.)

(e) Prove that $\mathbb{1}_{A\Delta B} = \mathbb{1}_A + \mathbb{1}_B - 2\,\mathbb{1}_A \cdot \mathbb{1}_B$ and conclude that $\left(\mathbb{1}_{A\Delta B}(x) = 1\right) \Leftrightarrow \left(\mathbb{1}_A(x) + \mathbb{1}_B(x) \text{ is odd}\right)$.

(f) Prove that $\forall A, B \subseteq X, \left(\left(\forall x \in X,\ \mathbb{1}_A(x) \le \mathbb{1}_B(x)\right) \Leftrightarrow A \subseteq B\right)$.

Conclude $\forall A, B \subseteq X, \left(\mathbb{1}_A = \mathbb{1}_B \Leftrightarrow A = B\right)$.

4. Let X be a non-empty set. Let $\mathcal{F}(X, \{0,1\})$ be the

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set which consists of functions from X to $\{0,1\}$. So for instance $f:\{a,b\} \to \{0,1\}$, $f(a)=1$, $f(b)=0$ is a function from $\{a,b\}$ to $\{0,1\}$. Hence $f \in F(\{a,b\},\{0,1\})$.

Let $\Theta : \mathcal{P}(X) \to F(X,\{0,1\})$, $\Theta(A) = \mathbb{1}_A$ where $\mathbb{1}_A$ is the characteristic function of A as defined in the previous problem.

Prove that Θ is a bijection.