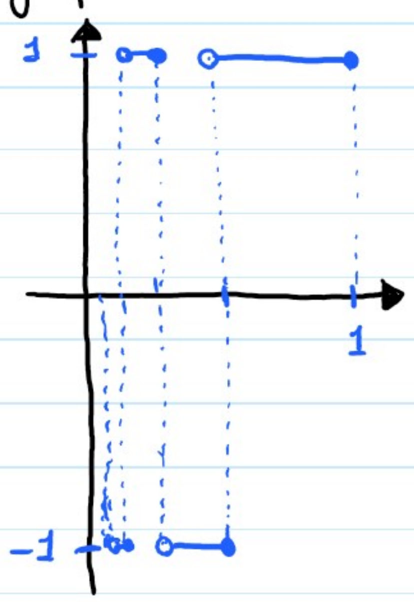
1. Let $f:(0,1] \to [-1,1]$, for any $k \in \mathbb{Z}^{\geq 0}$,

$$f(x) = (-1)^k \quad \text{if} \quad \frac{1}{2^{k+1}} < x \leq \frac{1}{2^k}.$$

So its graph looks like:



Prove that $\lim\limits_{x \to 0^+} f(x)$ does NOT exist.

Proof. We have to show $\forall L \in \mathbb{R}$, $\lim\limits_{x \to 0^+} f(x) \neq L$. I.e.

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall \delta > 0, \exists x, \; 0 < x < \delta \;\wedge\; |f(x) - L| \geq \varepsilon. \qquad \text{I}$$

We claim that in fact any $0 < \varepsilon < 1$, e.g. $\varepsilon = 1/2$, satisfies I. I.e. we prove the following:

$$\forall L \in \mathbb{R}, \forall \delta > 0, \exists x, \; 0 < x < \delta \;\wedge\; |f(x) - L| \geq 1/2. \qquad \text{II}$$

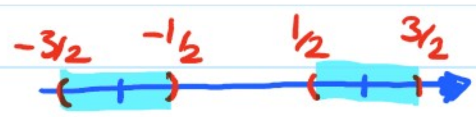
To prove II, first we show it is enough to prove the following:

$$\forall \delta > 0, \exists \; 0 < x_1, x_2 < \delta \;\wedge\; f(x_1) = 1 \;\wedge\; f(x_2) = -1 \qquad \text{III}$$

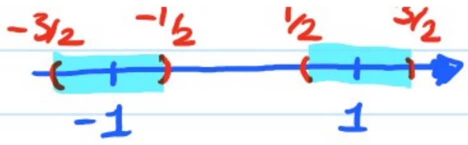
Suppose III holds. Then, for any $L \in \mathbb{R}$ and $\delta > 0$, let x_1 and x_2 be the numbers that satisfy III. Then we claim either x_1 satisfies II or x_2 does.

If NOT, then $|f(x_1) - L| < 1/2$ and $|f(x_2) - L| < 1/2$.

$$\Rightarrow \begin{cases} |1 - L| < 1/2 \\ |-1 - L| < 1/2 \end{cases} \Rightarrow \begin{cases} L > 1/2 \\ L < -1/2 \end{cases}, \text{ which is a contradiction.}$$



$\lfloor\, |-1-L| < 1/2 \,\rfloor \qquad \lfloor\, L < -1/2$



Proof of (III) Suppose k is a positive integer such that $\frac{1}{2^k} < \delta$.

Then $0 < \frac{1}{2^{k+1}} < \frac{1}{2^k} < \delta$ and $f\left(\frac{1}{2^k}\right) = (-1)^k$ and $f\left(\frac{1}{2^{k+1}}\right) = (-1)^{k+1}$

so $\frac{1}{2^k}$ and $\frac{1}{2^{k+1}}$ can be x_1 and x_2 (or x_2 and x_1).

(Notice that $\frac{1}{2^k} < \delta \Leftarrow \frac{1}{\delta} < 2^k \Leftarrow \log_2(1/\delta) < k$.

So it is enough to take an integer $k > \log_2(1/\delta)$).

2.(a) Prove or disprove: $\forall x \in \mathbb{R},\ ((\forall \varepsilon > 0,\ |x| \leq \varepsilon) \Rightarrow x = 0)$.

(b) Prove or disprove: $\forall x \in \mathbb{R}, \forall \varepsilon > 0,\ (|x| \leq \varepsilon \Rightarrow x = 0)$.

Solution. (a) It is true, and we prove it by contradiction. Suppose to the contrary that its negation holds:

$$\exists x \in \mathbb{R},\ (\forall \varepsilon > 0,\ |x| \leq \varepsilon) \wedge x \neq 0,$$

which is the same as:

$$\exists x \in \mathbb{R} \setminus \{0\},\ \forall \varepsilon > 0,\ |x| \leq \varepsilon. \quad \text{(I)}$$

Notice that, for $x \in \mathbb{R} \setminus \{0\}$, $0 < |x|$. So $0 < |x|/2 < |x|$, which contradicts (I).

(a) An alternative, fairly close, method is proving the contrapositive statement: $\forall x \in \mathbb{R},\ (x \neq 0 \Rightarrow (\exists \varepsilon > 0,\ |x| > \varepsilon))$

$$\equiv \forall x \in \mathbb{R} \setminus \{0\},\ \exists \varepsilon > 0,\ |x| > \varepsilon. \quad (*)$$

For a given $0 \neq x \in \mathbb{R}$, we need to find $\varepsilon > 0$ which satisfies $(*)$. As above, we know $\varepsilon = |x|/2$ satisfies $(*)$.

(b) It is false. We show that its negation is true.

$$\exists x \in \mathbb{R},\ \exists \varepsilon > 0,\ |x| < \varepsilon \wedge x \neq 0. \quad \text{(I)}$$

(I) is clearly true. For instance, $x = 1$, $\varepsilon = 2$ satisfy (I).

3. Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. $(x,y) \in A \times (B \cup C) \Leftrightarrow x \in A \wedge y \in B \cup C$

$$\Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$

$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$$

$$\Leftrightarrow (x,y) \in A \times B \vee (x,y) \in A \times C$$

$$\Leftrightarrow (x,y) \in (A \times B) \cup (A \times C).$$

4. (a) Find all possible $a \in \mathbb{R}$ such that

$$\exists x \in \mathbb{R},\ x^2 - 2x + a^2 = 0.$$

(b) Find all possible $a \in \mathbb{R}$ such that

$$\exists!\, x \in \mathbb{R},\ x^2 - 2x + a^2 = 0.$$

Solution. (a) We have to find all $a \in \mathbb{R}$ such that $x^2 - 2x + a^2 = 0$ has a real-valued solution.

$$x^2 - 2x + a^2 = 0 \Leftrightarrow x^2 - 2x + 1 = 1 - a^2$$

$$\Leftrightarrow (x-1)^2 = 1 - a^2$$

It has a real-valued solution if and only if $1 - a^2 \geq 0$

$\Leftrightarrow a^2 \leq 1 \Leftrightarrow |a| \leq 1 \Leftrightarrow -1 \leq a \leq 1$.

(b) $x^2 - 2x + a^2 = 0$ has a unique real-valued solution if and only if $(x-1)^2 = 1 - a^2$ has a unique real-valued solution. If $1 - a^2 < 0$, it has NO solution. If $1 - a^2 > 0$, it has two solutions. So $1 - a^2 = 0$, which implies $a = \pm 1$.

5. Prove that there are 2^n functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$.

Proof. We use induction on n.

Base. $n = 1$. There are 2 functions $f_1, f_2 : \{1\} \to \{0, 1\}$.

$f_1(1) = 0$ and $f_1(0) = 1$. And $2 = 2^1$.

Inductive step. For any positive integer k, Inductive step: For any positive integer k,

$$\left\{\begin{array}{l}\text{number of functions} \\ f:\{1,\dots,k\}\to\{0,1\} \\ \text{is } 2^k\end{array}\right\} \overset{?}{\Longrightarrow} \left\{\begin{array}{l}\text{number of functions} \\ f:\{1,2,\dots,k+1\}\to\{0,1\} \\ \text{is } 2^{k+1}.\end{array}\right.$$

For any function $f:\{1,2,\dots,k+1\}\to\{0,1\}$, we have that either $f(k+1)=0$ or $f(k+1)$.

<u>How many functions $f:\{1,2,\dots,k+1\}\to\{0,1\}$ are there such that $f(k+1)=0$</u> ?

Any such functions is uniquely determined by its values on $\{1,2,\dots,k\}$, and vice versa any function $g:\{1,\dots,k\}\to\{0,1\}$ can be uniquely <u>extended</u> to a function

$$f:\{1,\dots,k,k+1\}\to\{0,1\}$$

such that $f(k+1)=0$ (to be precise $f(i)=g(i)$ for $1\le i\le k$.). Hence the number of such functions by the induction hypothesis is 2^k.

<u>Similarly</u> the number of functions $f:\{1,\dots,k+1\}\to\{0,1\}$ such that $f(k+1)=1$ is 2^k.

Hence the number of functions $f:\{1,2,\dots,k+1\}\to\{0,1\}$ is $2^k+2^k=2^{k+1}$.

6. For $A\subseteq X$, the characteristic function $\mathbb{1}_A$ of A is

$$\mathbb{1}_A : X\to\{0,1\},\quad \mathbb{1}_A(x)=\begin{cases}1 & x\in A \\ 0 & x\in X\setminus A.\end{cases}$$

(a) Prove that $\mathbb{1}_{A\cap B}=\mathbb{1}_A\cdot\mathbb{1}_B$.

(b) Prove that $\mathbb{1}_{A^c}+\mathbb{1}_A=\mathbb{1}_X$.

(a) <u>Proof.</u>

$x\in A$	$x\in B$	$x\in A\cap B$	$\mathbb{1}_A(x)$	$\mathbb{1}_B(x)$	$\mathbb{1}_A(x)\cdot\mathbb{1}_B(x)$	$\mathbb{1}_{A\cap B}(x)$
T	T	T	1	1	1	1

(a) Proof.

$x\in A$	$x\in B$	$x\in A\cap B$	$\mathbb{1}_A(x)$	$\mathbb{1}_B(x)$	$\mathbb{1}_A(x)\cdot\mathbb{1}_B(x)$	$\mathbb{1}_{A\cap B}(x)$
T	T	T	1	1	1	1
T	F	F	1	0	0	0
F	T	F	0	1	0	0
F	F	F	0	0	0	0

So $\mathbb{1}_A(x)\cdot\mathbb{1}_B(x)=\mathbb{1}_{A\cap B}(x)$ for any x.

(b) Proof.

$x\in A$	$x\in A^c$	$\mathbb{1}_A(x)$	$\mathbb{1}_{A^c}(x)$	$\mathbb{1}_A(x)+\mathbb{1}_{A^c}(x)$
T	F	1	0	1
F	T	0	1	1

$\mathbb{1}_A(x)+\mathbb{1}_{A^c}(x)=1=\mathbb{1}_X(x)$ for any $x\in X$.

Corollary $\mathbb{1}_{A\cup B}=\mathbb{1}_A+\mathbb{1}_B-\mathbb{1}_A\cdot\mathbb{1}_B$.

Proof (Method 1)

$$
\begin{aligned}
\mathbb{1}_{A\cup B} &= 1-\mathbb{1}_{(A\cup B)^c} && (\text{part (b)})\\
&= 1-\mathbb{1}_{A^c\cap B^c}\\
&= 1-\mathbb{1}_{A^c}\cdot\mathbb{1}_{B^c} && (\text{part (a)})\\
&= 1-(1-\mathbb{1}_A)(1-\mathbb{1}_B) && (\text{part (b)})\\
&= 1-(1-\mathbb{1}_A-\mathbb{1}_B+\mathbb{1}_A\cdot\mathbb{1}_B)\\
&= \mathbb{1}_A+\mathbb{1}_B-\mathbb{1}_A\cdot\mathbb{1}_B.
\end{aligned}
$$

(Method 2)
Use a truth-table to prove it.

Exercise. Can you find $\mathbb{1}_{A\Delta B}$ in terms of $\mathbb{1}_A$ and $\mathbb{1}_B$?

Remark. You can see the similarities between Problem 5 and $|P(\{1,2,\cdots,n\})|=2^n$. In your next problem set, you will see how $\mathbb{1}_A$ can help us to understand the connection between problem 5 and $|P(\{1,\cdots,n\})|=2^n$ better.

between problem 5 and $|\mathcal{P}(\{1, \dots, n\})| = 2^n$ better.