Monday, November 9, 2015 8:56 PM

1. Write down the negation of the following statements:

(a) $\forall \varepsilon > 0, \exists \delta > 0, |x-1| < \delta \Rightarrow |x^2-1| < \varepsilon$.

(b) $\forall \varepsilon > 0, \forall x \in \mathbb{R}, \exists n \in \mathbb{Z}, |x-n| < \varepsilon$

(c) Let α be an irrational number, i.e. $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

$$\forall \varepsilon > 0, \forall x \in \mathbb{R}, \exists m, n \in \mathbb{Z}, |x - m - n\alpha| < \varepsilon.$$

(d) $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}^{>0}$, there is a unique pair (q, r) of integers such that

$$a = bq + r \quad \text{and} \quad 0 \leq r < b.$$

(you are not allowed to use $\nexists$.)

Solution. $\exists \cdots \forall \cdots P$ is similar to a winning game. So its negation is a losing game i.e. $\forall \cdots \exists \cdots (\neg P)$. And vice versa.

(a) Losing game $\rightsquigarrow$ winning game

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{R}, |x-1| < \delta \wedge |x^2-1| \geq \varepsilon.$$

(b) Losing game $\rightsquigarrow$ winning game

$$\exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall n \in \mathbb{Z}, |x-n| \geq \varepsilon.$$

(c) Losing game $\rightsquigarrow$ winning game

$$\exists \varepsilon > 0, \exists x \in \mathbb{R}, \forall m, n \in \mathbb{Z}, |x - m - n\alpha| \geq \varepsilon.$$

(c) $\neg(\exists!\, a\in A,\ P(a)) \equiv (\nexists\, a\in A,\ P(a)) \vee (\exists a_1,a_2\in A,\ a_1\neq a_2 \wedge P(a_1)\wedge P(a_2))$

$\equiv (\forall a\in A, \neg P(a)) \vee (\exists a_1,a_2\in A,\ a_1\neq a_2 \wedge P(a_1)\wedge P(a_2))$

(uniqueness fails if either no elements satisfy P or at least two elements satisfy P.)

$$\exists a\in\mathbb{Z}, \exists b\in\mathbb{Z}^{>0}, \Big[(\forall q,r\in\mathbb{Z},\ a\neq bq+r \vee r<0 \vee b\leq r)$$
$$\vee (\exists q_1,r_1,q_2,r_2\in\mathbb{Z}, (q_1,r_1)\neq(q_2,r_2) \wedge a=bq_1+r_1 \wedge 0\leq r_1<b \wedge a=bq_2+r_2 \wedge 0\leq r_2<b.)\Big]$$

---

2. (a) Prove or disprove: $\exists x\in\mathbb{R}, \forall y\in\mathbb{R},\ y^2 > 2015+x$

(b) Prove or disprove: $\exists x\in\mathbb{R}, \forall y\in\mathbb{R},\ y^3 > 2015+x$

(c) Prove or disprove: $\forall \varepsilon>0, \exists N\in\mathbb{Z}^{>0}, n\geq N \Rightarrow \frac{1000}{n}<\varepsilon.$

(For part (c), you are allowed to use the following:

$\forall x\in\mathbb{R}, \exists n\in\mathbb{Z},\ x<n.$)

Solution. For any function $f:\mathbb{R}\to\mathbb{R}$,

$$\exists x\in\mathbb{R}, \forall y\in\mathbb{R},\ f(y)>x$$

is true if and only if f has a lower bound, i.e. graph of f is above a line parallel to the x-axis.

(a) True. $x=-2016$ works:

$$\forall y\in\mathbb{R},\ y^2\geq 0>-1=2015-2016.$$

(b) False. We have to show the negation holds:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad y^3 \le 2015 + x.$$

$\forall x \in \mathbb{R}$, let $y = \sqrt[3]{2015+x}$, then $y^3 = 2015 + x$.

(c) True. (You are essentially proving that $\lim_{n\to\infty} \frac{1000}{n} = 0$.

In general, we say $\lim_{n\to\infty} a_n = L$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{Z}, \quad n \ge N \Longrightarrow |a_n - L| < \varepsilon.$$

In plain English, it says:

For any $\varepsilon > 0$, for large enough n (depending on ε), a_n gets ε-close to L.)

$\forall \varepsilon > 0$, we need to find $N \in \mathbb{Z}$ such that

$$n \ge N \Longrightarrow \frac{1000}{n} \le \varepsilon.$$

For $n \ge N \ge 1$ we have $\frac{1000}{n} \le \frac{1000}{N}$. So it is enough to find $N \in \mathbb{Z}^{\ge 1}$ such that $\frac{1000}{N} \le \varepsilon$.

So an integer $N \ge \frac{1000}{\varepsilon}$ is a good choice. And by hint we know there is such integer.

---

3. Prove that $\lim_{x \to 2} x^3 = 8$.

<u>Proof</u>. We have to prove: $\forall \varepsilon > 0, \exists \delta > 0, \; 0 < |x-2| < \delta \Longrightarrow |x^3 - 8| < \varepsilon$.

$$|x^3 - 8| = |x-2|\,|x^2+2x+4| < \delta\, |x^2+2x+4| \qquad \text{I}$$

$$|x^3-8| = |x-2|\,|x^2+2x+4| < \delta\,|x^2+2x+4| \qquad \text{(I)}$$

If $\delta \leq 1$, then $|x-2| \leq 1 \Rightarrow 1 \leq x \leq 3$

$$\Rightarrow \quad 0 < x^2+2x+4 \leq 9+(2)(3)+4$$
$$= 19. \qquad \text{(II)}$$

(I), (II) $\Rightarrow |x^3-8| \leq 19\,\delta$.

If $\delta \leq \varepsilon/19$, then $|x^3-8| \leq \varepsilon$ as we wished.

Hence $\delta = \min\{1, \varepsilon/19\}$ is a good choice.

---

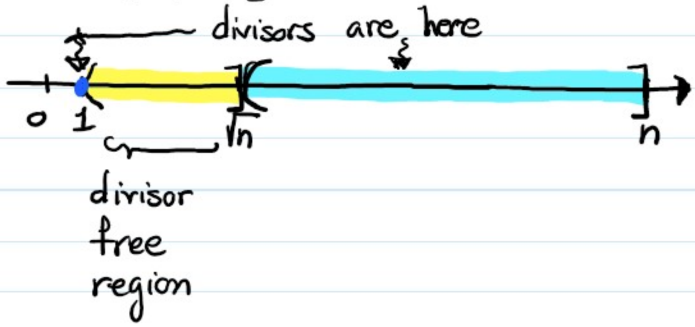
4. Prove that

$$\forall n \in \mathbb{Z}^{>1}, \left(\left(\nexists m \in \mathbb{Z},\ 1 < m \leq \sqrt{n} \wedge m \mid n\right) \Rightarrow n \text{ is prime.}\right.$$

<u>Proof.</u> First notice that

$\nexists m \in \mathbb{Z},\ 1 < m \leq \sqrt{n} \wedge m \mid n$ says that

the interval $(1, \sqrt{n}]$ has NO divisor of n.



Equivalently, $(m \mid n \wedge 1 < m) \Rightarrow \sqrt{n} < m$. (*)

Now we proceed by contradiction. I.e. we assume the negation holds and we reach to a contradiction.

Negation:

$\exists n \in \mathbb{Z}^{>1}$, $\circledast$ holds $\wedge$ n is NOT prime.

n is NOT prime $\Rightarrow$ $\exists k_1, k_2 \in \mathbb{Z}$ such that

$1 < k_1, k_2 \wedge n = k_1 \cdot k_2$ $\circledast\circledast$

$\Rightarrow 1 < k_1, k_2 \wedge k_1 | n \wedge k_2 | n$

(by $\circledast$) $\Rightarrow \sqrt{n} < k_1 \wedge \sqrt{n} < k_2$

$\Rightarrow \sqrt{n} \cdot \sqrt{n} < k_1 \cdot k_2$

(by $\circledast\circledast$) $\Rightarrow n < n$ which is a contradiction.

The last problem is related to sieve of Eratosthenes. This is an effective method to find all the primes $\leq X$.

- For $2 \leq i \leq \sqrt{X}$

  If i is NOT crossed out,

  cross out all the multiples of i starting from $2i$.

The remaining numbers are all the primes $\leq X$.

For instance to find all the primes ≤ 100, it is enough to cross out (non-trivial) multiples of $2, 3, 5, 7$.

2 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~

11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	93	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~