1. Prove that $\mathcal{P}(\{1,2,\dots,n\})$ has 2^n elements.

Proof. We use induction on n.

Base. $n=1$. We have to show $\mathcal{P}(\{1\})$ has $2^1=2$ elements.

$\mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ which has 2 elements.

Inductive step. For any $k \in \mathbb{Z}^{\geq 1}$, we need to show

$$|\mathcal{P}(\{1,2,\dots,k\})| = 2^k \Rightarrow |\mathcal{P}(\{1,2,\dots,k+1\})| = 2^{k+1}.$$

We split the subsets of $\{1,2,\dots,k+1\}$ into two groups:

Group 1. Those subsets X that do not contain $k+1$.

Group 2. Those subsets X that do contain $k+1$.

A subset X of $\{1,2,\dots,k+1\}$ does NOT contain $k+1$ if and only if X is a subset of $\{1,\dots,k\}$. So by the induction hypothesis there are 2^k such subsets.

A subset X of $\{1,2,\dots,k+1\}$ contains $k+1$ if and only if $X = X' \cup \{k+1\}$ where X' is a subset of $\{1,2,\dots,k\}$. Notice that $X' = X \setminus \{k+1\}$. So again by the induction hypothesis there are 2^k such subsets.

So overall $\{1,2,\dots,k+1\}$ has $2^k+2^k=2^{k+1}$ subsets.

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2. List the elements of $X=\{A\subseteq\{1,2,3\} \mid |A| \text{ is even}\}$

and the elements of $Y=\{A\subseteq\{1,2,3\} \mid |A| \text{ is odd}\}$.

<u>Solution</u>. $\mathcal{P}(\{1,2,3\})=\{\{\},\{1\},\{2\},\{1,2\},$
$\{3\},\{1,3\},\{2,3\},\{1,2,3\}\}$.

$|\{\}|=0$, $|\{1\}|=1$, $|\{2\}|=1$, $|\{1,2\}|=2$,

$|\{3\}|=1$, $|\{1,3\}|=2$, $|\{2,3\}|=2$, $|\{1,2,3\}|=3$

So $X=\{\{\},\{1,3\},\{2,3\},\{1,2\}\}$ and

$Y=\{\{1\},\{2\},\{3\},\{1,2,3\}\}$.

<u>Remark</u>. Let $X_n=\{A\subseteq\{1,2,\dots,n\} \mid$ the number of elements of A is even$\}$,
$Y_n=\{A\subseteq\{1,2,\dots,n\} \mid$ " " " " " " is odd$\}$.

Then $|X_n|=|Y_n|$. So combining by the 1^{st} problem, $|X_n|=|Y_n|=2^{n-1}$.

You can prove the above claim providing a "matching" between the elements of X_n and Y_n. Use problem 5 to show the following is a "matching" between elements of X_n and Y_n:

$$A\longmapsto A\,\Delta\,\{1\}.$$

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3. Find the truth-value of the following statements; justify your answer:

(a) There are No sets A and B such that

$A \in B \quad \wedge \quad A \subseteq B.$

(b) $\{\varnothing\} \subseteq \{1, 2, \{\varnothing\}\}$.

<u>Solution</u> (a) False. It is enough to find one example.

Let $A = \{1\}$ and $B = \{1, \{1\}\}$.

(b) False. $\varnothing \in \{\varnothing\} \ \wedge \ \varnothing \notin \{1, 2, \{\varnothing\}\}$

So $\{\varnothing\} \not\subseteq \{1, 2, \{\varnothing\}\}$.

<u>Remark</u>. For part (a), A can be any set. Then

$$\left.\begin{array}{l} A \in \{A\} \\ A \subseteq A \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} A \in \{A\} \cup A, \\ A \subseteq \{A\} \cup A. \end{array}\right.$$

Let $B = \{A\} \cup A$. Then $A \in B \ \wedge \ A \subseteq B$.

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4. Prove that, for any two sets A and B,

$$A \subseteq B \Longleftrightarrow A = A \cap B$$

<u>Proof</u>. ($\Longrightarrow$) We have to show $A = A \cap B$. So we need to show

$$A \subseteq A \cap B \quad \wedge \quad A \supseteq A \cap B.$$

- $x \in A \Longrightarrow x \in B$ since $A \subseteq B$. So

$$\begin{aligned} x \in A &\Longrightarrow x \in A \ \wedge \ x \in B \\ &\Longrightarrow x \in A \cap B. \end{aligned}$$

Therefore $A \subseteq A \cap B$. (I)

- $$\begin{aligned} x \in A \cap B &\Longrightarrow x \in A \ \wedge \ x \in B \\ &\Longrightarrow x \in A \end{aligned}$$

Thus $A \cap B \subseteq A$. (II)

• ①, ② $\Longrightarrow A = A\cap B$.

($\Longleftarrow$)

$$x\in A \Longrightarrow x\in A\cap B \qquad \text{since } A = A\cap B$$
$$\Longrightarrow x\in A \wedge x\in B$$
$$\Longrightarrow x\in B$$

So $A\subseteq B$.

Remark. The same argument as above shows that we always have $A\cap B\subseteq A$.

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5. (a) Prove that $A\triangle A=\varnothing$ and $A\triangle\varnothing = A$

(b) Prove that $A\triangle B = A\triangle C \Longrightarrow B=C$

Proof. (a) $A\triangle A = (A\setminus A)\cup(A\setminus A) = \varnothing\cup\varnothing=\varnothing$.

$A\triangle\varnothing = (A\setminus\varnothing)\cup(\varnothing\setminus A) = A\cup\varnothing = A$.

(b) $A\triangle B = A\triangle C \Longrightarrow A\triangle(A\triangle B) = A\triangle(A\triangle C)$

(by hint) $\Longrightarrow (A\triangle A)\triangle B = (A\triangle A)\triangle C$

(part (a)) $\Longrightarrow \varnothing\triangle B = \varnothing\triangle C$

(part (a)) $\Longrightarrow B=C$.

Remark. Part (a) shows $\varnothing$ for $\triangle$ is similar to 0 for $+$, and A for A and $\triangle$ is similar to $-a$ for a and $+$. The above argument is similar to the argument which says
$$a+b=a+c \Longrightarrow b=c.$$

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6 Use qu fiers to write down that $\Lambda\subset\mathbb{R}$ [NOT h ve

6. Use quantifiers to write down that $A \subseteq \mathbb{R}$ does NOT have a minimum.

<u>Solution.</u> In class we said A has minimum is equivalent to

$$\exists x \in A, \forall y \in A, \ x \leq y.$$

So its negation is

$$\forall x \in A, \exists y \in A, \ x > y.$$

(In plain English it says for any element of A there is a smaller one in A.)