Solutions 1.

Monday, October 12, 2015 3:12 PM

1. Construct truth tables for the following propositional forms.

(i) $P \wedge (Q \vee R)$ (iii) $\neg(P \Rightarrow Q)$

(ii) $(P \wedge Q) \vee (P \wedge R)$ (iv) $(\neg P) \wedge Q$

Solution

P	Q	R	$\neg P$	$Q \vee R$	$P \wedge Q$	$P \wedge R$	$P \Rightarrow Q$	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$	$\neg(P \Rightarrow Q)$	$(\neg P) \wedge Q$
T	T	T	F	T	T	T	T	T	T	F	F
T	T	F	F	T	T	F	T	T	T	F	F
T	F	T	F	T	F	T	F	T	T	T	F
T	F	F	F	F	F	F	F	F	F	T	F
F	T	T	T	T	F	F	T	F	F	F	T
F	T	F	T	T	F	F	T	F	F	F	T
F	F	T	T	T	F	F	T	F	F	F	F
F	F	F	T	F	F	F	T	F	F	F	F
1	2	3	4	5	6	7	8	9	10	11	12
			1	2,3	1,2	1,3	1,2	1,5	6,7	8	2,4

$\boxed{\text{add a column for } \underline{2}}$

$\neg Q$	$P \wedge (\neg Q)$
F	F
F	F
T	T
T	T
F	F
F	F
T	F
T	F
13	14

2 ↦ (column 13)

← 1, 13 (column 14)

2. Use the first problem to deduce that

(i) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (distributive law)

(ii) $\neg(P \Rightarrow Q) \equiv P \wedge (\neg Q)$

Solution. (i) Looking at columns 9 and 10, we see that they have the same truth values.

(ii) Looking at columns 11 and 14, we see that they have the same truth values.

3. Prove that for any positive real numbers a and b we have $\sqrt{ab} \geq \min\{a,b\}$.

Proof. We prove it by contradiction. Suppose to the contrary that for some positive real numbers a and b we have

$$\sqrt{ab} < \min\{a,b\} \Rightarrow \begin{cases} \sqrt{ab} < a \\ \sqrt{ab} < b \end{cases} \Rightarrow \sqrt{ab} \cdot \sqrt{ab} < a \cdot b$$

$$\Rightarrow ab < ab$$

which is a contradiction.

There are other methods, but in most of them you have to point out that you are using $|x| \leq |y| \iff x^2 \leq y^2$.

4. Prove that for any real numbers a and b we have

$$|a+b| \leq |a| + |b|.$$

(Hint. $x^2 \leq y^2 \iff |x| \leq |y|$ ① and $z \leq |z|$ ②.)

Solution (Backward argument)

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$$|a+b| \le |a|+|b| \Longleftarrow (|a+b|)^2 \le (|a|+|b|)^2 \qquad \text{(Hint ①)}$$

$$\Longleftarrow a^2+2ab+b^2 \le a^2+2|ab|+b^2$$

$$(|x|^2=x^2,\ |xy|=|x||z|)$$

$$\Longleftarrow 2ab \le 2|ab| \qquad (x<y \Rightarrow x+z<y+z)$$

$$\Longleftarrow \underbrace{ab \le |ab|} \qquad \left(\left.\begin{array}{l} x<y \\ 0<z \end{array}\right\} \Rightarrow xz<yz\right)$$

true because of Hint ② for $z=ab$.

5. Prove that for all integers n,

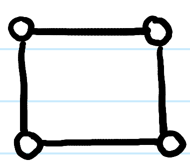
$$4(n^2+n+1)-3n^2$$

is a perfect square.

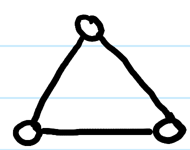
Solution. $4(n^2+n+1)-3n^2 = n^2+4n+4 = (n+2)^2$ and $n+2$ is integer.

6. We would like to color each circle in a way that two connected circles have different colors.
What is the minimum number of needed colors? Justify your answer.

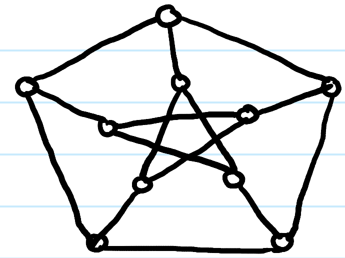
(i)



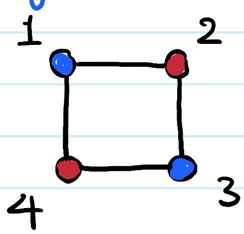
(ii)



(iii)

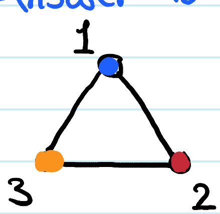


<u>Solution</u> (i) Answer is 2. We need to color the graph with 2 colors, and prove that we cannot do it with only 1 color.



Since 1 and 2 are connected, they cannot have the same color. So we need at least <u>2</u> colors.

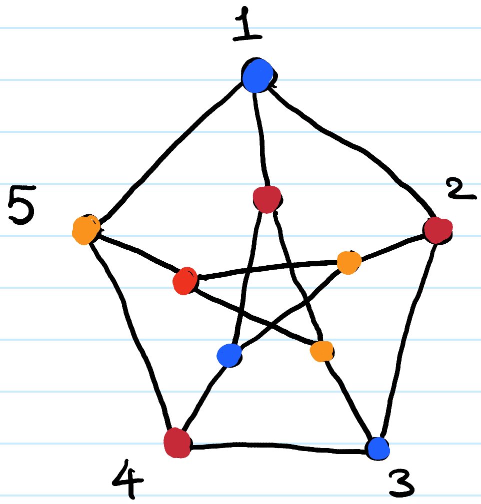
(ii) Answer is 3.



We prove by contradiction that this graph cannot be colored by <u>2</u> colors.

Suppose to the contrary that we can color it by <u>Blue</u> and <u>Red</u>. W.L.O.G we can assume 1 is colored <u>blue</u>. Since 2 and 3 are connected to 1, they cannot be blue. Since we have only two colors available, 2 and 3 should be red. This is a contradiction as 2 and 3 are connected and have the same colors.

(iii)



Answer is 3.

We prove by contradiction that this graph cannot be colored by two colors, say blue and red. W.L.O.G we can and will assume that 1 is colored <u>blue</u>. Since 2 and 5 are connected to 1, they cannot be blue. Since only two colors are allowed, 2 and 5 should be <u>red</u>.

Since 3 is connected to 2, 3 cannot be red
Since 4 " " " 5, 4 " " " $\Rightarrow$

3 and 4 are blue, which is a contradiction as 3 and 4 are connected.