1. (a) We use backward argument.

$$f(x) \le f(y) \Leftarrow \frac{2x+1}{x+1} \le \frac{2y+1}{y+1}$$

$(x+1, y+1 > 0)$ $\Leftarrow (2x+1)(y+1) \le (2y+1)(x+1)$

$\Leftarrow 2xy + 2x + y + 1 \le 2yx + 2y + x + 1$

$(a<b \Rightarrow a+c<b+c)$ $\Leftarrow x \le y$.

(b) We use induction on n.

Base $n=0$. We have to check $a_0 \le \frac{1+\sqrt{5}}{2}$.

Backward argument. $1 \le \frac{1+\sqrt{5}}{2} \Leftarrow 2 \le 1+\sqrt{5}$ $(c>0, a<b \Rightarrow ac<bc)$

$\Leftarrow 1 \le \sqrt{5}$ $(a<b \Rightarrow a+c<b+c)$

$\Leftarrow 1 \le 5$ $(|x| \le |y| \Leftrightarrow x^2 \le y^2)$

✓

Inductive Step. For any $k \in \mathbb{Z}^{\ge 0}$,

$$a_k \le t_0 \overset{?}{\Longrightarrow} a_{k+1} \le t_0.$$

$a_k \le t_0 \Rightarrow f(a_k) \le f(t_0)$ ($f$ is increasing.)

$\Rightarrow a_{k+1} \le t_0$. $(a_{k+1} = f(a_k) \wedge f(t_0) = t_0)$.

(c) We use induction on n.

Base. $n=0$. We have to check $a_0 \le a_1$.

Backward argument. $1 \le \frac{2+1}{1+1} = \frac{3}{2} \Leftarrow 2 \le 3$ ✓

Inductive step. For any $k \in \mathbb{Z}^{\ge 0}$,

$$a < a \overset{?}{\Longrightarrow} a < a$$

$$a_k \le a_{k+1} \overset{?}{\Longrightarrow} a_{k+1} \le a_{k+2}.$$

$$a_k \le a_{k+1} \Rightarrow f(a_k) \le f(a_{k+1}) \qquad (f \text{ is increasing.})$$

$$\Rightarrow a_{k+1} \le a_{k+2} \qquad (a_{k+1} = f(a_k) \wedge a_{k+2} = f(a_{k+1})).$$

(d) Since $\{a_n\}_{n=1}^{\infty}$ is a bounded, increasing sequence, $\lim_{n\to\infty} a_n = L$ exists. Moreover $L \ge a_1 = 1$ as $\{a_n\}_{n=1}^{\infty}$ is increasing.

Take the limit of both sides of $a_{n+1} = \dfrac{2a_n+1}{a_n+1}$ as $n \to \infty$. So we have

$$L = \frac{2L+1}{L+1} \qquad (\text{as } L+1 \ne 0).$$

$$\Rightarrow L^2 - L - 1 = 0$$

$$\left.\begin{array}{l} \Rightarrow L = \dfrac{1+\sqrt{5}}{2} \text{ or } \underbrace{\dfrac{1-\sqrt{5}}{2}}_{\text{is negative}} \\ L \ge 1 \end{array}\right\} \Rightarrow L = \frac{1+\sqrt{5}}{2}.$$

$\blacksquare$

2. We use induction on $n$.

<u>Base</u>. $n=1$. We have to check $1^2 = \dfrac{(1)(1+1)(2+1)}{6}$ which is true.

<u>Inductive step</u>. For any $k \in \mathbb{Z}^{\ge 1}$,

$$1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \overset{?}{\Longrightarrow} 1^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$1^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad (\text{induction hypothesis})$$

$$1^2+\cdots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \qquad \text{(induction hypothesis)}$$

$$= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]$$

$$= \frac{(k+1)}{6}\left(2k^2+7k+6\right)$$

$$= \frac{(k+1)}{6}(k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}. \qquad \blacksquare$$

3. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. We use strong induction on $n$ to prove $f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$ for any $n \in \mathbb{Z}^{\geq 0}$.

<u>Base</u>. $n=0$. We have to check $f_0 = \frac{1}{\sqrt{5}}(\alpha^0 - \beta^0)$, which is clear as LHS $= 0 =$ RHS.

<u>Strong induction step.</u> For any $k \in \mathbb{Z}^{\geq 0}$,

For any integer $0 \leq m \leq k$, $f_m = \frac{1}{\sqrt{5}}(\alpha^m - \beta^m) \overset{?}{\Longrightarrow} f_{k+1} = \frac{1}{\sqrt{5}}(\alpha^{k+1} - \beta^{k+1})$.

<u>If $k=0$</u>, then $f_{k+1} = f_1 = 1$ and

$$\frac{1}{\sqrt{5}}(\alpha^1 - \beta^1) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1 \quad \checkmark$$

<u>If $k \geq 1$</u>, then $f_{k+1} = f_k + f_{k-1}$

(by strong induct. hypothesis.) $$= \frac{1}{\sqrt{5}}(\alpha^k - \beta^k) + \frac{1}{\sqrt{5}}(\alpha^{k-1} - \beta^{k-1})$$

$k \qquad k \quad k$

hypothesis.) $\quad \sqrt{5} \qquad \sqrt{5}$

$$= \frac{1}{\sqrt{5}}\left[(\alpha^k+\alpha^{k-1}) - (\beta^k+\beta^{k-1})\right]$$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k-1}(\alpha+1) - \beta^{k-1}(\beta+1)\right]$$

Roots of $x^2-x-1=0$ are $\frac{1\pm\sqrt{5}}{2}$ $\Rightarrow \begin{cases}\alpha^2=\alpha+1\\ \beta^2=\beta+1\end{cases}$

$$= \frac{1}{\sqrt{5}}\left[\alpha^{k-1}\cdot\alpha^2 - \beta^{k-1}\cdot\beta^2\right]$$

$$= \frac{1}{\sqrt{5}}\left(\alpha^{k+1} - \beta^{k+1}\right). \qquad \blacksquare$$

4. (a) We use induction on $n$.

<u>Base</u>. $n=1$. We have to check $b_1 \overset{?}{=} \frac{f_2}{f_1}$.

LHS $= 1 =$ RHS.

<u>Inductive step</u>. For any $k \in \mathbb{Z}^+$

$$b_k = \frac{f_{k+1}}{f_k} \overset{?}{\Longrightarrow} b_{k+1} = \frac{f_{k+2}}{f_{k+1}}.$$

$$b_{k+1} = 1 + \frac{1}{b_k} = 1 + \frac{1}{f_{k+1}/f_k} = 1 + \frac{f_k}{f_{k+1}} = \frac{f_{k+1}+f_k}{f_{k+1}} = \frac{f_{k+2}}{f_{k+1}}.$$

Induction hypothesis

(b) $$b_{n+1} - b_n = \frac{f_{n+2}}{f_{n+1}} - \frac{f_{n+1}}{f_n}$$ (by part (a))

$$= \frac{f_{n+2}\cdot f_n - (f_{n+1})^2}{f_{n+1}\cdot f_n}$$

$$= \frac{(-1)^{n+1}}{f_{n+1}\cdot f_n}$$ (in class we proved that

$f \cdot f \quad f^2 - (1)^n$ )

$f_{n+1} \cdot f_n$ $\qquad$ $f_{n+1} \cdot f_{n-1} - f_n^{\,2} = (-1)^n \cdot)$

5. We use strong induction on n.

<u>Base.</u> $n=34$. We need to show that 34 can be written in the form of $5x+9y$ for some $x, y \in \mathbb{Z}^{\geq 0}$.

$$34 = 5 \times 5 + 9 \times 1$$

<u>Strong inductive step.</u> For any integer $k \geq 34$,

$$\left.\begin{array}{l}\text{For any integer } 34 \leq m \leq k, \\ \quad m = 5x+9y \\ \text{for some } x, y \in \mathbb{Z}^{\geq 0}\end{array}\right\} \overset{?}{\Longrightarrow} \begin{array}{l} k+1 = 5x+9y \\ \text{for some } x, y \in \mathbb{Z}^{\geq 0}.\end{array}$$

Let $m = (k+1) - 5 = k-4$.

<u>If $34 \leq m$</u>, then $34 \leq m \leq k$, and so, by the strong induction hypothesis, $m = 5x+9y$ for some $x, y \in \mathbb{Z}^{\geq 0}$.

$$\Rightarrow k+1 = m+5$$
$$= 5(x+1) + 9y$$

and $x+1, y \in \mathbb{Z}^{\geq 0}$.

<u>If $m < 34$</u>, then $35 \leq k+1 < 39$. So

$$k+1 = 35 = 5 \times 7 + 9 \times 0$$

or $k+1 = 36 = 5 \times 0 + 9 \times 4$

or $k+1 = 37 = 5 \times 2 + 9 \times 3$

$k+1 = 37 \quad = 5 \times 2 + 9 \times 3$

or

$k+1 = 38 \quad = 5 \times 4 + 9 \times 2$

So in any case we have that $k+1 = 5x + 9y$ for some $x, y \in \mathbb{Z}^{\geq 0}$.

$\blacksquare$