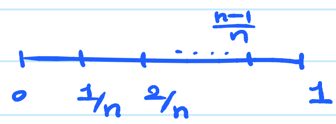
Tuesday, November 24, 2015 12:51 PM

1.(I) Suppose $a_0, a_1, \dots, a_n \in [0,1]$. Prove that for some $0 \le i \ne j \le n$, $\quad |a_i - a_j| \le \frac{1}{n}$.

(Hint. Use pigeonhole principle, and 

.)

(II) Let $\alpha \in \mathbb{R}$. Prove that,

$$\forall n \in \mathbb{Z}^+,\ \exists m \in \mathbb{Z},\ \exists k \in \mathbb{Z},$$

$$0 < m \le n \ \wedge\ |m\alpha - k| \le \frac{1}{n}.$$

(Hint. Let $a_i = i\alpha - \lfloor i\alpha \rfloor$ and use part (I).)

(III) Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Prove that <u>for infinitely many</u> pairs of integers (m,k) we have

$$\left|\alpha - \frac{k}{m}\right| \le \frac{1}{m^2}.$$

(Hint.

Suppose there are only finitely many such pairs:

$(m_1, k_1), \dots, (m_s, k_s)$. Since $\alpha \notin \mathbb{Q}$, $\min\{|m_i\alpha - k_i| \,|\, 1 \le i \le s\} \ne 0$.

So for some $n \in \mathbb{Z}^+$, $\frac{1}{n} < \min\{|m_i\alpha - k_i| \,|\, 1 \le i \le s\}$.

Now use part (II), and notice $\frac{1}{n} \le \frac{1}{m}$ if $0 < m \le n$.)

2.(I) Prove that $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ -1 & \text{if } x \notin \mathbb{Z}. \end{cases}$

(II) Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$.

(Hint. Case 1. $\lfloor 2x \rfloor$ is even $\Rightarrow$ $\exists k \in \mathbb{Z}$, $2k \le 2x < 2k+1 \Rightarrow$
$k \le x < k + 1/2 \ \wedge\ k + 1/2 \le x + 1/2 < k+1$.

Case 2. $\lfloor 2x \rfloor$ is odd $\Rightarrow$
$\exists k \in \mathbb{Z}$, $2k+1 \leq 2x < 2k+2 \Rightarrow$
$k+\frac{1}{2} \leq x < k+1 \wedge k+1 \leq x+\frac{1}{2} < k+\frac{3}{2}$.)

3. (I) Prove that if $f: A_1 \to A_2$ and $g: B_1 \to B_2$ are bijections,

then $h: A_1 \times B_1 \to A_2 \times B_2$, $h(a,b) = (f(a), g(b))$

is a bijection.

(II) Prove that, if $A_1, \ldots, A_n$ are enumerable sets, then

$A_1 \times \cdots \times A_n$ is enumerable.

(Hint. Use induction on n, and the fact that

we proved in class: $\mathbb{Z}^+ \times \mathbb{Z}^+$ is enumerable.)

4. In this exercise you are allowed to use the fact that

any positive integer has a unique binary representation, i.e.

$\forall n \in \mathbb{Z}^+, \exists! \; m_1, \ldots, m_k \in \mathbb{Z}^{\geq 0}, \quad 0 \leq m_1 < m_2 < \cdots < m_k$

and $\quad n = 2^{m_k} + 2^{m_{k-1}} + \cdots + 2^{m_1}$.

(I) Prove that $\{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\}$ is enumerable.

(Hint. Let $f: \{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\} \to \mathbb{Z}^+$,

$f(\{m_1, \ldots, m_k\}) = 2^{m_1} + \cdots + 2^{m_k}$.)

(II) Prove that there is no surjection

$$g: \{X \subseteq \mathbb{Z}^{\geq 0} \mid X \text{ is finite}\} \to \mathcal{P}(\mathbb{Z}^{\geq 0}),$$

where $\mathcal{P}(\mathbb{Z}^{\geq 0})$ is the power set of $\mathbb{Z}^{\geq 0}$.

(Hint. Cantor.)

5. Determine if the following functions are injective or surjective.

Justify your answers.

(I) $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, $f((a,b)) = 3a - 2b$.

(II) Let $A \subseteq X$, and $\ell: P(X) \to P(X)$, $\ell(B) = A \Delta B$.

(Hint. What is $\ell \circ \ell(B)$?)

(III) Let Y be a non-empty subset of X, and

$r: P(X) \to P(Y)$, $r(B) = Y \cap B$.

6. Suppose $f: X \to X$ is a function and $f \circ f = f$.

Prove that, $\forall x \in X$, $x \in \mathrm{Im}(f) \iff f(x) = x$.