Saturday, October 17, 2015 6:44 PM

1. Let $f(x) = 1 + \dfrac{1}{1+\frac{1}{x}} = \dfrac{2x+1}{x+1}$ for positive real number x.

(a) Prove that $f(x)$ is increasing, i.e. for any real numbers x_1 and x_2,

$$x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2).$$

Let $a_0 = 1$, $a_{n+1} := f(a_n)$.

(b) Prove that for any non-negative integer n,

$$a_n \le \frac{1+\sqrt{5}}{2} =: t_0$$

[Notice that $\dfrac{1}{t_0} = \dfrac{\sqrt{5}-1}{2} = t_0 - 1$. So

$$f(t_0) = 1 + \frac{1}{1+\frac{1}{t_0}} = 1 + \frac{1}{t_0} = t_0.$$]

(c) Prove that $\{a_n\}_{n=0}^{\infty}$ is an increasing sequence, i.e. for any non-negative integer n, $a_n \le a_{n+1}$.

(d) Prove that $\lim_{n\to\infty} a_n = \dfrac{1+\sqrt{5}}{2}$.

[Using a similar technique one can show that

$b_0 = 2 = 1 + \frac{1}{1}$, $b_{n+1} = f(b_n)$ defines a decreasing sequence which converges to $\frac{1+\sqrt{5}}{2}$. Altogether we have $1+\cfrac{1}{1+\cfrac{1}{1+\cdots}} = \frac{1+\sqrt{5}}{2}$.

This is an example of a continued fraction.]

2. Prove that for any positive integer n,

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Let $\{f_n\}_{n=0}^{\infty}$ be the Fibonacci sequence, i.e.

$f_0=0$, $f_1=1$, $f_{n+1}=f_n+f_{n-1}$ for any positive integer n.

Prove that for any positive integer n,

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

4. Let $b_1=1$, $b_{n+1} = 1+\frac{1}{b_n}$ for any positive integer n.

So we get the following initial terms:

$$1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$$

(a) Prove that for any positive integer n,

$$b_n = \frac{f_{n+1}}{f_n},$$

where $\{f_n\}_{n=0}^{\infty}$ is the Fibonacci sequence.

(b) Prove that for any positive integer n,

$$b_{n+1}-b_n = \frac{(-1)^{n+1}}{f_n f_{n+1}}.$$

$$\ _{n+1} \quad \ _n = \frac{\quad}{f_n f_{n+1}}.$$

[Hint. Use one of the properties of the Fibonacci sequence that I proved in class.].

[Remark. Problem 1 together with 4.a implies that

$$\lim_{n\to\infty} \frac{f_{2n+1}}{f_{2n}} = \frac{\sqrt{5}+1}{2}.$$

Using 4.b we get $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{\sqrt{5}+1}{2}$. ⊛

- You can use Problem 3 to give another proof for ⊛.]

5. (Postage stamp problem) Prove that any postage greater than 34 can be obtained by stamps of denominations 5 and 9.

[Hint ⊙ You need to show for any integer $n \geq 34$ there are non-negative integers x and y such that

$$n = 5x + 9y.$$

① Use strong induction on n.

② $34 = 5 \times 5 + 9$,

$35 = 5 \times 7$,

$36 = 9 \times 4$,

7 9

$38 = 5 \times 4 + 9 \times 2$.]