Math 109 Midterm 2 Review

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1 Language of set theory.

1. List all subsets of $\{1, 2, \{1\}, \{1, 2\}\}$.
2. Show that if $(A \cup C) \subseteq (A \cup B)$ and $(A \cap C) \subseteq (A \cap B)$, then $C \subseteq B$.
3. Define the complement of a subset A of X as $A^c = \{a \in X |\ a \notin A\}$. Let A, B be two subsets of X. Prove the following De Morgan's Laws for sets.
   (a) $(A \cap B)^c = A^c \cup B^c$
   (b) $(A \cup B)^c = A^c \cap B^c$
4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
5. Let A and B be non-empty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.
6. Let $A \triangle B := (A \setminus B) \cup (B \setminus A)$. Prove that $A \triangle B = (A \cup B) \setminus (A \cap B)$.

2 Quantifiers.

1. Prove or disprove: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $\forall z \in \mathbb{R}, x + y = z$.
2. Show that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}$ with $n \geq N$,
$$\frac{n}{n^2 - 1} < \epsilon.$$
3. Prove that
$$\forall \varepsilon > 0, \exists\, \delta > 0,\ 0 < |x - 1| < \delta \Rightarrow |x^2 - 1| < \varepsilon.$$
4. For each part, prove or disprove the statement.
   (a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, -x^4 < y$.
   (b) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, -x^4 < y$.
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, -x^3 < y$.
   (d) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, -x^3 < y$.
5. An integer $n > 1$ is called a 2-almost prime if it is product of at most two prime factors. For instance $2, 3, 5, 6, 7, 9, 10$ are 2-almost primes, but $8, 12, 16, 18$ are NOT 2-almost primes. Let $\mathcal{P}_2$ be the set of all the 2-almost primes. Prove that
$$\forall n \in \mathbb{Z}^{>1}, n \notin \mathcal{P}_2 \Rightarrow \exists\ m_1, m_2, m_3 \in \mathbb{Z}^{>1}, n = m_1 \cdot m_2 \cdot m_3.$$
6. Suppose that $A \subseteq \mathbb{Z}$. Write the following statement entirely in symbols using quantifiers. Then write the negative of the statement using symbols. Finally, give an example of a set A for which the statement is true, and an example of a set A for which the statement is false. The statement is:

   "There is a greatest number in the set A."

3 Functions.

1. Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = x^2 - 1$.

   (a) Find the functions $f \circ f$, $f \circ g$, $g \circ f$, and $g \circ g$.

   (b) Suppose we changed the codomain of f to be $\mathbb{R}^{\geq 0}$. Which of the functions in part (a) would still be defined?

   (c) List all elements of the set $\{x \in \mathbb{R} | \ f(g(x)) = g(f(x))\}$.

2. Given $Y \subseteq Z$, define the characteristic function of Y as the function $\mathbb{1}_Y : Z \to \{0, 1\}$ where

$$\mathbb{1}_Y(z) = \begin{cases} 1 & \text{if } z \in Y \\ 0 & \text{if } z \notin Y \end{cases}$$

   Suppose that A and B are subsets of Z.

   (a) Prove that $\mathbb{1}_{A \cap B} = \mathbb{1}_A(z)\mathbb{1}_B(z)$. That is, prove that the characteristic function of the intersection $A \cap B$ is the product of the characteristic functions of A and B.

   (b) Find a subset $C \subseteq Z$ whose characteristic function is given by $\mathbb{1}_C = \mathbb{1}_A(z) + \mathbb{1}_B(z) - \mathbb{1}_A(z)\mathbb{1}_B(z)$.