Monday, November 23, 2015 11:03 AM

In the previous lecture we defined the concept of <u>cardinality</u>.

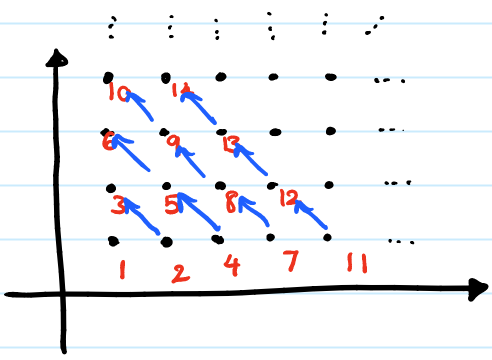
- For two sets A and B we say $|A| = |B|$ if there is a <u>bijection</u> $f: A \longrightarrow B$.

<u>Ex.</u> $|\mathbb{Z}| = |\mathbb{Z}^+| = |\mathbb{Z}^{\geq 0}|$.

<u>Ex.</u> $|\mathbb{Z}^+| = |\mathbb{Z}^+ \times \mathbb{Z}^+|$, i.e. there is a bijection

$$f: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+$$

Here is the function:



we are "enumerating" the integer points. Since each point is counted <u>once</u> and exactly <u>once</u>, it gives us a <u>surjective</u> and <u>injective</u> function

$$f: \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+ \times \mathbb{Z}^+.$$

<u>Definition</u>.. X is called <u>enumerable</u> if $\exists$ a bijection

$$f: \mathbb{Z}^+ \longrightarrow X.$$

. X is called <u>countable</u> if X is either finite or enumerable.

Is there any set that is NOT countable?

<u>Theorem</u>. (Cantor) For any set X, there is <u>no</u> surjection

$$f: X \longrightarrow P(X).$$

(<u>Warning</u>. The importance of this theorem is for infinite sets. Please do NOT say "since $2^n > n$".)

<u>Proof</u>. Suppose to the contrary that there is such surjection

Proof. Suppose to the contrary that there is such surjection

$$f : X \longrightarrow \mathcal{P}(X).$$

Let $A = \{x \in X \mid x \notin f(x)\} \subseteq X$.

Since f is surjective and $A \in \mathcal{P}(X)$, $A = f(a)$ for some $a \in X$.

Case 1. $a \in A \Rightarrow a \notin f(a) \Rightarrow a \notin A$ which is a contradiction.

Case 2. $a \notin A \Rightarrow a \in f(a) \Rightarrow a \in A$ which is a contradiction. ∎

Corollary $\mathcal{P}(\mathbb{Z}^+)$ is NOT countable.

There is a hierarchy of infinite sets.

Division algorithm $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}^+, \exists! (q, r) \in \mathbb{Z} \times \mathbb{Z}$,

(1) $a = bq + r$

(2) $0 \leq r < b$

(q is called the quotient and r is called the remainder.)

Lemma. $\forall x \in \mathbb{R}, \exists! n \in \mathbb{Z}$, $n \leq x < n+1$. $(\circledast)$

Proof. Existence. If $x \geq 0$, then let $n = \max \underbrace{\{k \in \mathbb{Z} \mid 0 \leq k \leq x\}}_{\text{is a finite set}}$.

Then $n+1 > n = \max\{k \in \mathbb{Z} \mid 0 \leq k \leq x\}$ implies that $x < n+1$.

- If $x < 0$, then let $m = \max\{k \in \mathbb{Z} \mid 0 \leq k < -x\}$.

So $m < -x \leq m+1$

$\Rightarrow -(m+1) \leq x < -m$. So $n = -(m+1)$ satisfies $(\circledast)$

Uniqueness. $\left.\begin{array}{l} n_1 \leq x < n_1 + 1 \\ \end{array}\right\} \Rightarrow \left\{ n_1 < n_2 + 1 \right. \Rightarrow n_1 - n_2 < 1 \Big\} \Rightarrow n_1 = n_2.$

Uniqueness: $$\left.\begin{array}{l} n_1 \le x < n_1+1 \\ n_2 \le x < n_2+1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} n_1 < n_2+1 \Rightarrow n_1-n_2<1 \\ n_2<n_1+1 \Rightarrow -1<n_1-n_2 \\ \qquad\qquad\qquad n_1-n_2\in\mathbb{Z} \end{array}\right\} \Rightarrow n_1=n_2.$$ ∎

Definition $\forall x\in\mathbb{R}$, the integer given in the above lemma is called the integer part of x. It is denoted by $\lfloor x\rfloor$.

So $\lfloor x\rfloor \le x < \lfloor x\rfloor+1$,

$\lfloor 0.9\rfloor = 0$; $\lfloor -0.5\rfloor = -1$;