Friday, November 20, 2015 10:58 AM

In the previous lecture we were in the middle of the proof of:

<u>Theorem.</u> $f: X \to Y$ is a bijection $\Leftrightarrow$ f is invertible.

<u>Proof.</u> ($\Leftarrow$) was proved in the previous lecture.

($\Rightarrow$) We need to find $g: Y \to X$ such that

$f \circ g = I_Y$ and $g \circ f = I_X$.

We have to decide what $x \in X$ to assign to $y \in Y$.

$\forall y \in Y$, since f is onto, $\exists\, x \in X$ such that

$$f(x) = y.$$

Since we want to have $(g \circ f)(x) = x$ we <u>have</u> to let $g(y)$ be x.

$\boxed{\text{Is it a function?}}$ i.e. do we have a <u>clear</u> assignment?

$$\left.\begin{array}{l} f(x_1) = y \\ f(x_2) = y \end{array}\right\} \Rightarrow f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

as f is injective.

So it is a function.

$\boxed{\text{Do we have } g \circ f = I_X \text{ ?}}$

$(g \circ f)(x) = g(f(x)) = x$ because of the way we defined g.

$\boxed{\text{Do we have } f \circ g = I_Y \text{ ?}}$

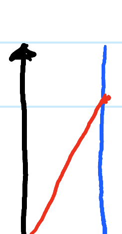
$$(f \circ g)(y) = f(g(y)) = f(x) = y$$

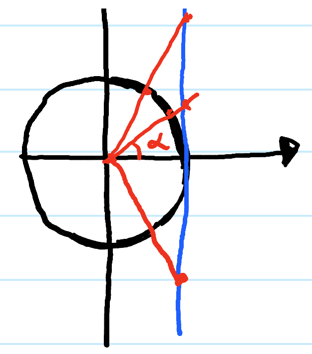
$\boxed{g(y) = x \text{ if } y = f(x)}$

$\blacksquare$

<u>Ex.</u> Is there a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$?

<u>Solution</u>. Yes, $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.





Ex. Is there a bijection between $\mathbb{Z}$ and $\mathbb{Z}^+$?

Solution. Yes! $f: \mathbb{Z} \to \mathbb{Z}^+$, $f(k) = \begin{cases} 2(k+1) & \text{if } k \geq 0, \\ -2k-1 & \text{if } k < 0. \end{cases}$

f is injective. $f(k_1) = f(k_2) \overset{?}{\Longrightarrow} k_1 = k_2$.

Let $n = f(k_1) = f(k_2) \in \mathbb{Z}^+$.

Case 1. n is even.

$\Rightarrow k_1, k_2 \geq 0$ and $n = 2(k_1+1) = 2(k_2+1)$

$\Rightarrow k_1 + 1 = k_2 + 1 \Rightarrow k_1 = k_2$.

Case 2. n is odd.

$\Rightarrow k_1, k_2 < 0$ and $n = -2k_1 - 1 = -2k_2 - 1$

$\Rightarrow -2k_1 = -2k_2 \Rightarrow k_1 = k_2$.

f is surjective $\forall n \in \mathbb{Z}^+, \exists k \in \mathbb{Z}, f(k) = n$ (?)

Case 1. $2 \mid n \Rightarrow$ we need to find $k \in \mathbb{Z}^{\geq 0}$ s.t.

$2(k+1) = n$.

$\Rightarrow k = \frac{n}{2} - 1$.

Since $2 \mid n$ and $n \in \mathbb{Z}^+$, $\frac{n}{2} - 1 \in \mathbb{Z}^{\geq 0}$.

$\Rightarrow n = f\left(\frac{n}{2} - 1\right)$.

Case 2. $2 \nmid n \Rightarrow$ we need to find $k \in \mathbb{Z}^{<0}$ s.t.

$-2k - 1 = n$

$\Rightarrow k = -\frac{n+1}{2}$.

Since $2 \nmid n \Rightarrow n = 2l+1 \Rightarrow n+1 = 2l+2$

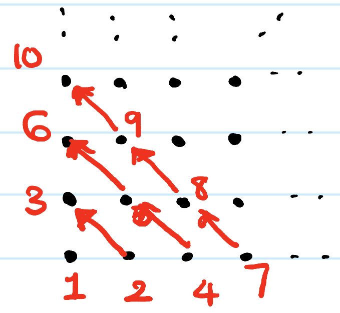
$\Rightarrow \frac{n+1}{2} \in \mathbb{Z}$.

$n \geq 1 \Rightarrow \frac{n+1}{2} \geq 1 \Rightarrow -\frac{n+1}{2} \in \mathbb{Z}^{<0}$

So $n = f\left(-\frac{n+1}{2}\right)$. ∎

Ex. Is there a bijection between $\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$?

Solution. Yes



Here is a bijective function $\mathbb{Z}^+ \times \mathbb{Z}^+ \xrightarrow{f} \mathbb{Z}^+$.

$f(1,1)=1,\ f(2,1)=2,\ f(1,2)=3, \dots$

Proposition. ① $\exists$ a bijection $X \xrightarrow{f} Y \iff \exists$ a bijection $Y \xrightarrow{g} X$

② $\left.\begin{array}{l} \exists \text{ a bijection } X \xrightarrow{f} Y \\ \exists \text{ a bijection } Y \xrightarrow{g} Z \end{array}\right\} \Rightarrow \exists$ a bijection $X \xrightarrow{h} Z$

Proof. ① ($\Rightarrow$) f is invertible and (f^{-1}) is a bijection.

② $g \circ f$ is a bijection. $\blacksquare$

Definition. We say two sets have the same cardinality if there is a bijection between them.

Definition X is called a finite set if there is a bijection $f: \{1,2,\dots,n\} \to X$ for some $n \in \mathbb{Z}^{\geq 0}$.

Theorem If there is an injection $f: \{1,2,\dots,n\} \to \{1,2,\dots,m\}$, then $n \leq m$.

Theorem. If there is a surjection $f: \{1,2,\dots,n\} \to \{1,2,\dots,m\}$, then $n \geq m$.

Corollary. If there is a bijection $f: \{1,2,\dots,n\} \to \{1,2,\dots,m\}$, then $n = m$.

The first theorem is called pigeonhole principle. Another way of formulating it is: if $n > m$, then any function

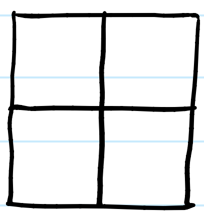
$$f: \{1,2,\dots,n\} \to \{1,2,\dots,m\}$$

is NOT injective.

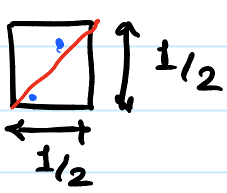
$\underline{n}$ pigeons $>$ $\underline{m}$ pigeonholes $\Rightarrow$ at least two pigeons should share a pigeonhole.

<u>Ex</u>. Suppose P_1, P_2, P_3, P_4, and P_5 are five points in a unit square. Then the distance of (at least) two of them is at most $1/\sqrt{2}$.

<u>Solution</u>.



By Pigeonhole principle, at least two points P_i and P_j are in the same small square.



So $P_iP_j \le$ diam. of small square

$$= \sqrt{(1/2)^2 + (1/2)^2} = \frac{1}{\sqrt{2}}. \quad \blacksquare$$