Monday, November 16, 2015 11:13 AM

In the previous lecture we defined <u>injective</u> and <u>surjective</u> functions.

Today we will explore properties of such functions.

<u>Proposition</u> Let $f: X \to Y$ and $g: Y \to Z$ be two functions.

If f and g are injective, then $g \circ f$ is injective.

<u>Proof</u>. $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

(Since g is injective) $\Rightarrow f(x_1) = f(x_2)$

(Since f is injective) $\Rightarrow x_1 = x_2$. ■

<u>Proposition</u> $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two functions.

If f and g are surjective, then $g \circ f$ is surjective.

<u>Proof</u>. We have to show $\mathrm{Im}(g \circ f) = Z$. i.e.

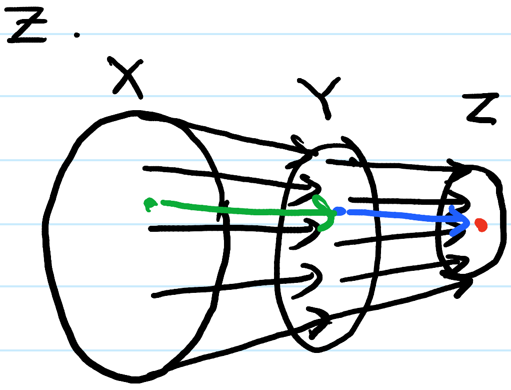
$\forall z \in Z, \exists x \in X, (g \circ f)(x) = z$.

. $\forall z \in Z$, for some $y \in Y$, $g(y) = z$ as g is surjective.

. Since f is surjective, for some $x \in X$,

$f(x) = y$.

So $(g \circ f)(x) = g(f(x)) = g(y) = z$. ■



<u>Proposition</u>. $X \xrightarrow{f} Y$, $Y \xrightarrow{g} X$ are two functions.

If $g \circ f = I_X$, then f is injective and g is surjective.

<u>Proof</u>. We have already proved that f is injective. Now we would like to prove g is surjective.

$\forall x \in X, (g \circ f)(x) = I_X(x) \Rightarrow g(f(x)) = x$

So $f(x) \in Y$ is a "good choice".

(Recall, we had to show

$$\forall x \in X, \exists y \in Y, \quad g(y) = x.) \qquad \blacksquare$$

<u>Definition</u>. $X \xrightarrow{f} Y$ is called <u>bijective</u> if it is <u>injective</u> and <u>surjective</u>.

- We say $Y \xrightarrow{g} X$ is an <u>inverse</u> of $X \xrightarrow{f} Y$ if $g \circ f = I_X$ and $f \circ g = I_Y$.
- $X \xrightarrow{f} Y$ is called <u>invertible</u> if it has an inverse.

<u>Proposition</u>. If $X \xrightarrow{f} Y$ is an invertible function, it has a <u>unique</u> inverse.

<u>Proof</u>. Suppose $Y \xrightarrow{g_1} X$ and $Y \xrightarrow{g_2} X$ are two inverses of f. So

$$\begin{aligned} g_1 &= I_X \circ g_1 \\ &= (g_2 \circ f) \circ g_1 \\ &= g_2 \circ (f \circ g_1) \\ &= g_2 \circ I_Y \\ &= g_2. \qquad \blacksquare \end{aligned}$$

Inverse of $X \xrightarrow{f} Y$ is denoted by $f^{(-1)}: Y \to X$.

<u>Proposition</u>. $X \xrightarrow{f} Y$ is invertible $\iff$ it is bijective.

<u>Proof</u>. $(\Rightarrow)$ For some $Y \xrightarrow{g} X$ we have

$$\left.\begin{cases} g \circ f = I_X \implies f \text{ is injective } \circledast \\ f \circ g = I_Y \implies f \text{ is surjective } \circledast \end{cases}\right\} \implies f \text{ is bijective.}$$

$(\Leftarrow)$ <u>Claim1</u> $\forall y \in Y, \exists! x \in X, f(x) = y$.

<u>Proof of claim1</u> If not, then

$$\exists y \in Y, \ (\nexists x \in X, f(x) = y) \vee$$

$$(\exists x_1, x_2 \in X, x_1 \neq x_2 \wedge f(x_1) = f(x_2) = y)$$

The first one cannot hold as f is surjective, and the second one cannot hold as f is injective. $\square$

Let $Y \xrightarrow{g} X$, $g(y) = x$ be the unique element of X such that $f(x) = y$.

So $f(g(y)) = y \Rightarrow f \circ g = I_Y$.

<u>Claim 2</u> If $f \circ g = I_Y$ and f is a bijection, then

$$g \circ f = I_X.$$

<u>Proof.</u> $$\left.\begin{array}{l} f(g \circ f(x)) = (f \circ g)(f(x)) = f(x) \\ f \text{ is injective} \end{array}\right\} \Rightarrow (g \circ f)(x) = x$$

$$\Rightarrow g \circ f = I_X. \qquad \blacksquare$$