**Ex.** Is there a function $f: \mathbb{R} \to \mathbb{R}$ s.t. $\operatorname{Im}(f) = \mathbb{Z}$?

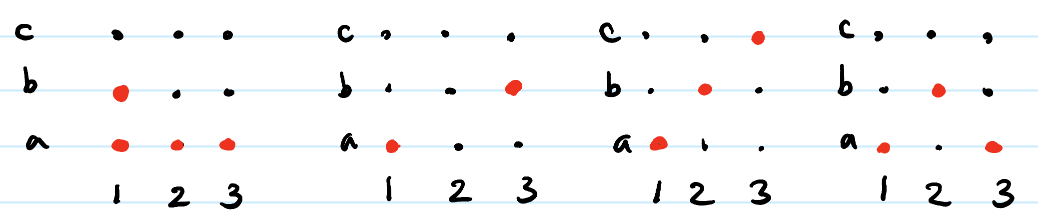
**Solution.** Yes, let $f(x) = \begin{cases} x & x \in \mathbb{Z} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$

**Ex.** Is there a function $f: \mathbb{R} \to \mathbb{R}$ s.t. $\operatorname{Im}(f) = \mathbb{R} \setminus \mathbb{Z}$?

**Solution.** Yes, let $f(x) = \begin{cases} x & x \in \mathbb{R} \setminus \mathbb{Z}. \\ 1/2 & x \in \mathbb{Z}. \end{cases}$

Graph of $f: A \to B$ is $G_f := \{(a, f(a)) \mid a \in A\} \subseteq A \times B$.

**Ex.** Which one is graph of a function $f: \{1,2,3\} \to \{a,b,c\}$?



NOT, it does not assign a unique value to 1 | NOT, it does NOT assign any value to 2 | yes | yes (though $f(1) = f(3) = a$.)

**Ex.** Suppose $G_f = \{(1,1), (2,3), (4,1)\}$. Find the domain and the image of $f$.

**Solution.** The first components give us the domain and the second components give us the image:

domain of $f = \{1, 2, 4\}$

$\operatorname{Im}(f) = \{1, 3\}$.

**Definition.** $f: A \to B$ is called <u>1-1</u> or <u>injective</u> if

$$\forall a_1, a_2 \in A, \left( f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \right).$$

**Ex.** $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x + 1$ is injective.

$f$ $f$

Pf. $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1 \Rightarrow x_1 = x_2$.

**Ex.** $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is NOT injective.

**Pf.** $f(1) = 1 = f(-1) \wedge 1 \neq -1$.

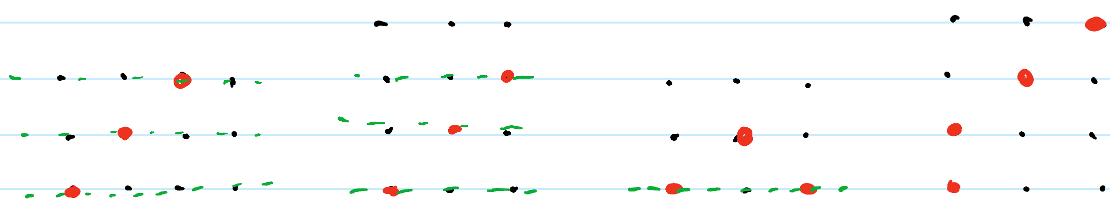
**Ex.** $f: \mathbb{R}^{>0} \to \mathbb{R}$, $f(x) = x^2$ is injective.

**Pf.** $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$

$\Rightarrow x_1 = x_2$ or $x_1 = -x_2$

$\left(\begin{array}{l}\text{Since } x_1 > 0 \\ \text{and } x_2 > 0\end{array}\right) \Rightarrow x_1 = x_2$.

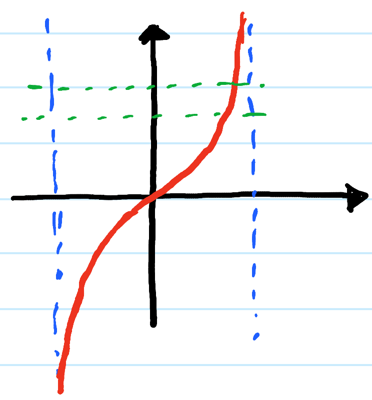
**Ex.** Which one is a 1-1 function?



NOT function | 1-1 function | is function but not 1-1 | NOT function

Ex. $f: (-\pi/2, \pi/2) \to \mathbb{R}$, $f(x) = \tan(x)$

is an injection.



**Ex.** $\left.\begin{array}{l} f: A \to B, \\ g: B \to A, \\ \text{Suppose } g \circ f = I_A \end{array}\right\} \Rightarrow f$ is injective.

**Pf.** $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow I_A(x_1) = I_A(x_2)$

$\Rightarrow x_1 = x_2.$ ∎

<u>Definition</u>. $f : A \to B$ is called <u>onto</u> or <u>surjective</u> if

$$\mathrm{Im}(f) = B.$$

Alternatively: $\forall b \in B, \exists a \in A, f(a) = B.$

<u>Ex.</u> In the above examples:

- $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is surjective
- $\mathbb{R} \to \mathbb{R}$
  $x \mapsto x^2$ is NOT surjective.