In mathematics/life, we often need to talk about <u>multiparameters</u> of a single object.

. For instance, netflix makes a "profile vector" of you:

what genre you like, what your age is, what your sex is, ...

. You look at nutritional facts of a meal:

calories, vitamins, minerals, ...

In order to put all these datas together, we use <u>n-tuple</u>.

<u>Definition</u>. For two sets A and B, the set of all the pairs (a,b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B.

$$A \times B = \{(x,y) \mid x \in A, \; y \in B\}.$$

. Order of the components is important.

<u>Ex</u>. $A = \{1,2\}$, $B = \{a,b\}$. List the elements of $A \times B$.

<u>Solution</u>. $A \times B = \{(1,a), (2,a), (1,b), (2,b)\}$.

<u>Ex</u>. Find $\left(\{0,1,2\} \times \{0,3\}\right) \cap \left(\{0,3\} \times \{0,1,2\}\right)$.

<u>Solution</u>. $\{0,1,2\} \times$

Solution. $\{0,1,2\} \times \{0,3\} = \{(0,0),(0,3),(1,0),(1,3),(2,0),(2,3)\}$

$\{0,3\} \times \{0,1,2\} = \{(0,0),(0,1),(0,2),(3,0),(3,1),(3,2)\}$

$\Longrightarrow$ intersection $= \{(0,0)\}$.

Lemma. $(A\times B)\cap(C\times D) = (A\cap C)\times(B\cap D)$.

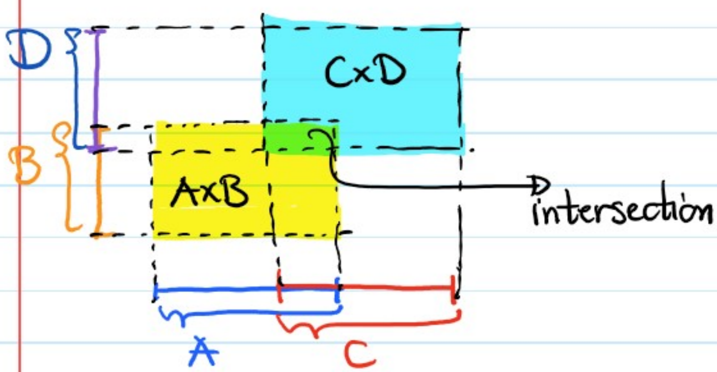
Proof. $(x,y)\in (A\times B)\cap(C\times D) \iff (x,y)\in A\times B \wedge (x,y)\in C\times D$

$\iff x\in A \wedge y\in B \wedge x\in C \wedge y\in D$

$\iff (x\in A \wedge x\in C) \wedge (y\in B \wedge y\in D)$

$\iff x\in A\cap C \wedge y\in B\cap D$

$\iff (x,y)\in(A\cap C)\times(B\cap D)$. ■



Ex. / Warning Prove or disprove:

$$(A\times B)\cup(C\times D) = (A\cup C)\times(B\cup D).$$

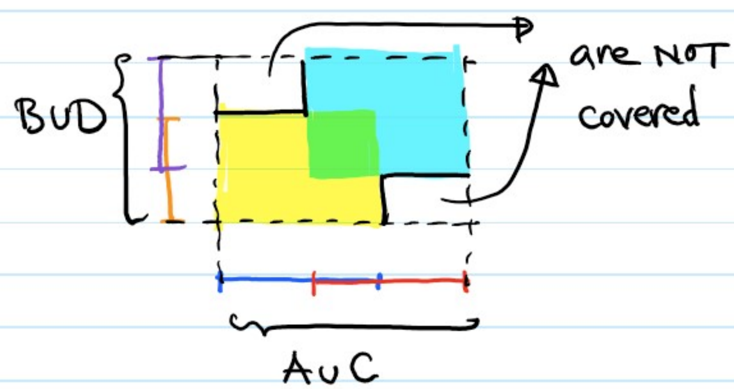
Solution. False.

$A=\{1\}$

$C=\{2\}$

$B=\{3\}$

$D=\{4\}$

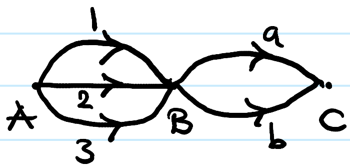


$\Longrightarrow$ $A\times B \cup C\times D = \{(1,3),(2,4)\}$

$(A\cup C)\times(B\cup D) = \{(1,3),(1,4),(2,3),(2,4)\}$. ■

. Number of elements of $A\times B$ is mn if A has m

Ex.



in how many ways can we go from A to C?

We can match each possible path with the elements of $\{1,2,3\} \times \{a,b\}$. So there are 6 possible ways to go to C from A.

Remark. $A \times \emptyset = \emptyset = \emptyset \times A$ for any set A.

You have seen the importance of functions.

Definition (rough) $f: X \longrightarrow Y$ or $X \xrightarrow{f} Y$

X: domain, Y: co-domain

rule of assigning.

$x \longmapsto f(x)$

Ex. $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$ is NOT a function

since it does NOT assign any value to $\underline{0}$.

We can fix it by changing its domain: $f: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$, $f(x) = 1/x$

is a function.

Ex. $f: \mathbb{R} \longrightarrow \mathbb{R}^{>0}$, $f(x) = x^2$ is NOT a function

since the value that it assigns to $\underline{0}$ is

NOT in the co-domain.

We can fix it by changing either the domain or the co-domain:

$$\left.\begin{array}{ll} f_1 : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}^{>0}, & f_1(x)=x^2 \\ f_2 : \mathbb{R} \longrightarrow \mathbb{R}^{\geq 0}, & f_2(x)=x^2 \end{array}\right\} \text{ are functions.}$$

<u>Definition</u>. Two functions $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ are equal if $X_1 = X_2$ and $Y_1 = Y_2$ and $\forall x \in X_1$, $f_1(x) = f_2(x)$.

<u>Ex</u>. $f : \mathbb{R} \longrightarrow \mathbb{R}^{\geq 0}$, $f(x)=x^2$
and $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(x)=x^2$ are NOT equal as they have different co-domains.