Today we do one example on induction.

Theorem. Let S_n be the of ordered finite sequences of positive integers that sum to n. For example

$$S_1 = \{(1)\}, \quad S_2 = \{(1,1), (2)\},$$

$$S_3 = \{(1,1,1), (1,2), (2,1), (3)\}.$$

Then the number of elements in S_n is 2^{n-1}.

In other words, there are 2^{n-1} ways to express n as an ordered sum of positive integers.

Proof. We use induction on n.

Base. $|S_1| \overset{?}{=} 2^{1-1}$

As we have seen, $S_1 = \{(1)\}$. Hence $|S_1| = 1 = 2^{1-1}$.

Inductive step. For any positive integer k,

$$|S_k| = 2^{k-1} \overset{?}{\Longrightarrow} |S_{k+1}| = 2^k.$$

Assuming that the theorem is true for some natural number k; that is, assume that S_k has 2^{k-1} elements.

We show that each element of S_k generates two elements of S_{k+1}.

Let $(a_1, a_2, \dots, a_m)$ be an element of S_k. By definition of S_k, we know that $a_1 + a_2 + \cdots + a_m = k$.

We generate two elements of S_{k+1} as follows.

The first one is generated by increasing the last term of the sequence by 1 to yield $(a_1, a_2, \dots, a_{m-1}, a_m + 1)$. This is an element of S_{k+1} since

$$a_1 + \cdots + a_{m-1} + (a_m + 1) = a_1 + \cdots + a_m + 1 = k+1.$$

The other element is generated by appending a 1 at the end of the sequence to yield $(a_1, a_2, \dots, a_m, 1)$. This is also an element of S_{k+1}.

Next we show that each element of S_{k+1} was generated from one element of S_k. Let $(b_1, \dots, b_m)$ be an element of S_{k+1}.

- If $b_m = 1$, then simply eliminate b_m from the sequence. In this case $(b_1, \dots, b_m)$ was generated by $(b_1, \dots, b_{m-1}) \in S_k$.
- If $b_m > 1$, then decrease the last entry of the sequence by 1. In this case, $(b_1, \dots, b_m)$ was generated by

$(b_1, b_2, \dots, b_{m-1}, b_m - 1) \in S_k$ (Note that $b_m - 1$ is a positive integer because $b_m > 1$.)

Since every element of S_k generates two elements is S_{k+1}, S_{k+1} has twice as many elements as S_k. By the inductive hypothesis, there are 2^{k-1} elements in S_k. Thus, there are $2 \times 2^{k-1} = 2^k$ elements is S_{k+1}. $\blacksquare$