Friday, October 30, 2015 11:06 AM

In the previous lecture we said what a winning game (P) and a losing game (N) are.

P : $\exists$ a move which makes it N.

N : $\forall$ move we get P.

Let's look at the definition of <u>limit</u>.

In calculus, you have seen $\lim_{x \to a} f(x) = L$.

Here is its formal definition:

$$\forall \varepsilon > 0, \exists \delta > 0, \; 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

It is like a <u>losing game</u> for any "move" of the 1st player ($\varepsilon$) the second player has a "move" ($\delta$).

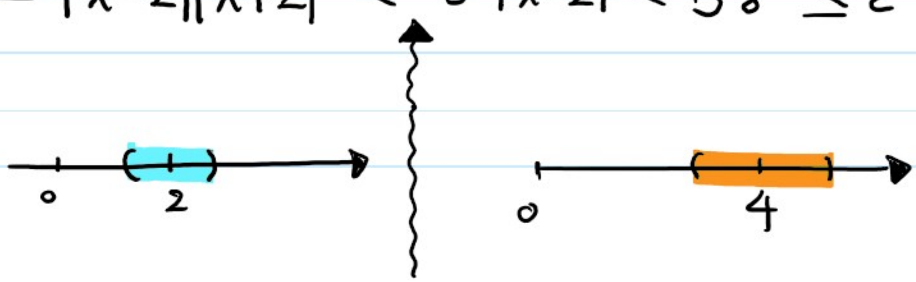
<u>Ex.</u> Prove that $\lim_{x \to 2} x^2 = 4$.

<u>Proof.</u> <u>$\forall \varepsilon > 0$</u> we need to show that $\exists \delta > 0$ s.t.

$$0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon.$$

$\boxed{\text{if } \delta \leq \varepsilon/5}$

$$|x^2 - 4| = |x-2||x+2| < 5|x-2| < 5\delta \leq \varepsilon$$




$$\boxed{\text{if } \delta<1 \Rightarrow |x-2|<1 \Rightarrow |x+2|<5.}$$

So $\delta = \min\{1, \varepsilon/5\}$ is a "good move", i.e.

$$\forall \varepsilon>0, \quad 0<|x-2|<\min\{1, \varepsilon/5\} \Longrightarrow |x^2-4|<\varepsilon. \qquad \blacksquare$$

**Ex.** Prove that $\displaystyle\lim_{x\to 2} \sqrt{x} = \sqrt{2}$.

**Proof.** $\forall \varepsilon>0$ we need to show that $\exists \delta>0$ s.t.

$$0<|x-2|<\delta \Longrightarrow |\sqrt{x}-\sqrt{2}|<\varepsilon.$$

$$|\sqrt{x}-\sqrt{2}| = |\sqrt{x}-\sqrt{2}|\cdot\frac{|\sqrt{x}+\sqrt{2}|}{|\sqrt{x}+\sqrt{2}|} = \frac{|x-2|}{|\sqrt{x}+\sqrt{2}|} \le \frac{\delta}{|\sqrt{x}+\sqrt{2}|}$$

$$\boxed{\begin{aligned} \text{if } \delta<1 &\Longrightarrow |x-2|<1 \\ &\Longrightarrow 1<x<3 \\ &\Longrightarrow 1<\sqrt{x}<\sqrt{3} \\ &\Longrightarrow 1+\sqrt{2}<\sqrt{x}+\sqrt{2}<\sqrt{3}+\sqrt{2} \\ &\Longrightarrow \frac{1}{|\sqrt{x}+\sqrt{2}|}<\frac{1}{1+\sqrt{2}}<1. \end{aligned}} \longrightarrow \le \delta$$

$$\le \varepsilon$$

$$\boxed{\text{if } \delta\le\varepsilon}$$

So $\delta = \min\{1, \varepsilon\}$ is a "good move". I.e. we have proved

$$\forall \varepsilon>0, \quad 0<|x-2|<\min\{1,\varepsilon\} \Longrightarrow |\sqrt{x}-\sqrt{2}|<\varepsilon. \qquad \blacksquare$$

The best way to make sure that we have understood this definition

is by considering <u>its negation</u>: $\lim_{x\to a} f(x) \neq L$

(it is a "winning game"; $1^{st}$ player has a "winning move" $\varepsilon_0$)

$$\exists\, \varepsilon > 0,\ \forall\, \delta > 0,\ \exists x,\ 0<|x-a|<\delta \ \wedge\ |f(x)-L| > \varepsilon_0 .$$

More interesting is the case when $\lim_{x\to a} f(x)$ does NOT exist.

$$\forall L,\ \exists\, \varepsilon>0,\ \forall\, \delta>0,\ \exists x,\ 0<|x-a|<\delta \ \wedge\ |f(x)-L| > \varepsilon_0 .$$

<u>Ex.</u> Prove that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does NOT exist.

we will show this on Wednesday.