In the previous lecture we said how we can say $A \subseteq \mathbb{R}$ has a minimum.

<u>Ex.</u> Use quantifiers to say a nonempty subset A of $\mathbb{R}$ is bounded.

<u>Solution</u>. $\exists\, m, M \in \mathbb{R},\ \forall\, a \in A,\quad m \leq a \leq M.$

<u>Ex.</u> Prove or disprove that any bounded non-empty subset $A \subseteq \mathbb{R}$ has a minimum.

<u>Solution</u>. We disprove it by proving that $(0,1)$ is bound, but it does not have a minimum.

<u>Bounded</u>. $\forall\, a \in (0,1),\quad 0 \leq a \leq 1 \implies$ it is bounded.

<u>No minimum</u>. Suppose to the contrary that it has a min.

$\implies \exists\, x \in (0,1),\ \forall y \in (0,1),\quad x \leq y.$ ⊛

$$x \in (0,1) \implies 0 < x$$

$$\left.\begin{aligned} &\implies 0 < x/2 \\ &\implies x/2 < x/2 + x/2 \end{aligned}\right\} \implies 0 < x/2 < x$$

$$\implies x/2 \in (0,1) \wedge x/2 < x$$

which contradicts ⊛. ∎

In order to get a better understanding of quantifiers and the

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a preference of their order. Let's play.

- Each time a player is supposed to say one of the numbers

  $$1, 2, 3, 4, 5$$

  A player wins if the mentioned numbers add up to 30.

- After a few rounds we make a <u>conjecture</u> on

  <u>winning cases</u> and <u>losing cases</u>.

$\mathcal{P}$ • A game is a <u>winning game</u> if the first player <u>has</u> a winning move <u>i.e</u> changes the game into a <u>losing game</u>.

N • A game is a <u>losing game</u> if no matter what the first player does the 2^{nd} player has a winning move.

<u>Using quantifiers</u>:

$\mathcal{P}$: $\exists$ <u>a move for player A</u>, $\forall$ <u>move of player B</u>,

A could win

N : $\forall$ <u>move of player A</u>, $\exists$ <u>move of player B</u>,

B could win.

<u>Alternatively</u>. Game is $\mathcal{P} \Longrightarrow \exists$ a move which makes it N.

Game is N $\Longrightarrow \forall$ move makes it $\mathcal{P}$.

<u>Ex</u>. In a game each player is supposed to say one of

the numbers 1, 2, 3, 4, 5. A player wins if the mentioned numbers add up to $\underline{\underline{n}}$.

Find all the $\underline{n\text{'s}}$ s.t. the above game is a losing game.

<u>Solution</u>. (In class we have to conjecture what the answer is:

n's that are multiples of 6.)

We use strong induction to show, for $n \in \mathbb{Z}^+$,

$$G(n) \text{ is } N \iff 6 \mid n.$$

<u>Base</u>. $G(1)$ is clear P as the 1^{st} player just says 1.

<u>Strong inductive step</u>. For any $k \in \mathbb{Z}^+$,

$$1 \le m \le k, \quad \left.\begin{array}{l} 6 \mid m \Rightarrow G(m) \text{ is } N \\ 6 \nmid m \Rightarrow G(m) \text{ is } P \end{array}\right\} \overset{?}{\Longrightarrow} \begin{array}{l} 6 \mid k+1 \Rightarrow G(k+1) \text{ is } N. \\ 6 \nmid k+1 \Rightarrow G(k+1) \text{ is } P. \end{array}$$

<u>Pf</u>. • $6 \mid k+1 \Rightarrow \left.\begin{array}{l} 6 \nmid (k+1)-1 \\ 6 \nmid (k+1)-2 \\ \vdots \\ 6 \nmid (k+1)-5 \end{array}\right\} \Rightarrow$ after the first move we get $G((k+1)-i)$ which is P by induction hyp.

$\Rightarrow G(k+1)$ is N.

• $6 \nmid k+1 \overset{?}{\Rightarrow} \left.\begin{array}{l} k+1 = 6q+r \\ 0 \le r < 6 \end{array}\right\} \Rightarrow$ if the first player takes out the remainder, we get $G(6q)$ which is N

$\Rightarrow G(k+1)$ is $\perp$. $\blacksquare$