Induction

Monday, October 12, 2015 11:03 AM

In the previous lecture we said what the induction principle is.

If $\boxed{P(1)}$ is true, and

$\boxed{\text{for any positive integer } k,\ P(k) \Rightarrow P(k+1)}$ is true,

then $\boxed{\text{for any positive integer } n,\ P(n) \text{ is true}}$.

Ex. Prove that, for any positive integer n,

$$1+3+\cdots+(2n-1)=n^2.$$

Proof. We use induction on n.

Base of induction. For $n=1$, the left hand side is 1, and the right hand side is 1. ✓

Inductive step. We have to show

$$1+3+\cdots+(2k-1)=k^2 \overset{?}{\Longrightarrow} 1+3+\cdots+(2k-1)+[2(k+1)-1] = (k+1)^2$$

$$1+3+\cdots+(2k-1)+[2(k+1)-1] = k^2+[2k+2-1]$$

Induction hypothesis

$$= k^2+2k+1$$

$$= (k+1)^2.$$

Hence, by the induction principle, for any positive integer n,

$$1+3+\cdots+(2n-1)=n^2.$$ ∎

$$1+3+\cdots+(2n-1)=n^2. \qquad \blacksquare$$

Ex. Let $a_1=\sqrt{2}$ and $a_{n+1}=\sqrt{2+a_n}$. Prove that for any positive integer n, $a_n<2$.

Proof. We use induction on n.

Base of induction. For $n=1$, we need to show

$$\sqrt{2}<2.$$

Backward argument. $\sqrt{2}<2 \Longleftarrow 2<4$ $\quad (|x|\le|y| \Leftrightarrow x^2\le y^2.)$

$\Longleftarrow 0=2-2<4-2=2$ $\quad (a<b \Rightarrow a-c<b-c)$

$\Longleftarrow 0<1$ $\quad \left(\begin{matrix} a<b \\ c<d \end{matrix}\right\} \Rightarrow a+c<b+d)$

Inductive step. We have to show

$$a_k<2 \overset{?}{\Rightarrow} a_{k+1}<2.$$

Pf of the inductive step. (backward argument)

$a_{k+1}<2 \Longleftarrow \sqrt{2+a_k}<2$

$\Longleftarrow 2+a_k<4$ $\quad (|x|\le|y| \Leftrightarrow x^2\le y^2)$

$\Longleftarrow \underbrace{a_k<2}_{\text{induction hypothesis.}}$ $\quad (a<b \Rightarrow a-c<b-c)$

$\blacksquare$

Ex. Prove that the above sequence is strictly increasing, i.e.

for any positive integers n, $a_n < a_{n+1}$.

Proof. We use induction on n.

Base of induction. For $n=1$, we need to show $a_1 < a_2$.

Backward argument. $a_1 < a_2 \Longleftarrow \sqrt{2} < \sqrt{2+\sqrt{2}}$

$\Longleftarrow 2 < 2+\sqrt{2}$ $\quad (|x| \leq |y| \Leftrightarrow x^2 \leq y^2)$

$\Longleftarrow 0 < \sqrt{2}$ $\quad (a<b \Rightarrow a-c<b-c)$

you can stop here

$\Longleftarrow 0 < 2$ $\quad (|x| \leq |y| \Leftrightarrow x^2 \leq y^2)$

$\Longleftarrow 0 < 1$ $\quad \left(\begin{matrix} a<b \\ c<d \end{matrix}\right\} \Rightarrow a+c<b+d.)$

Inductive step. We have to show

$$a_k < a_{k+1} \overset{?}{\Longrightarrow} a_{k+1} < a_{k+2}$$

Backward argument

$$a_{k+1} < a_{k+1} \Longleftarrow \sqrt{2+a_k} < \sqrt{2+a_{k+1}}$$

$\Longleftarrow 2+a_k < 2+a_{k+1}$ $\quad (|x| \leq |y| \Leftrightarrow x^2 \leq y^2.)$

$\Longleftarrow \underbrace{a_k < a_{k+1}}_{\text{induction hypothesis.}}$ $\quad (a<b \Rightarrow a-c<b-c)$

∎

Ex. Prove that $\lim_{n\to\infty} a_n = 2$.

Proof. By the previous examples, $\{a_n\}_{n=1}^{\infty}$ is a bounded and increasing sequence. So $\lim_{n\to\infty} a_n = L$ exists.

increasing sequence. So $\lim_{n\to\infty} a_n = L$ exists.

Taking the limit of both sides:

$$a_{n+1} = \sqrt{2+a_n}$$

we have $\quad L = \sqrt{2+L} \implies L^2 - L - 2 = 0$

$$\implies L = 2 \text{ or } L = -1$$

$L=-1$ is NOT possible as a_n's are positive. Hence $L=2$.

$\blacksquare$

(We have also defined $\sum_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} 2$, and mentioned $\prod_{i=1}^{n} 2 = 2^n$.)