# 2 is prime! And Induction

Friday, October 9, 2015 11:00 AM

In the previous lecture we were half-way through the proof of

**Theorem.** For any integers m and n,

$$mn \text{ is odd} \iff m \text{ is odd } \underline{\text{and}}\ n \text{ is odd.}$$

We proved ($\Leftarrow$) using proof-by-contradiction method.

One of you asked a subtle question She said the contrary assumption says: either m is even or n is even

we also know mn is odd.

Is it not enough to give an example for m and n to show the above cannot hold and get a contradiction?

$\boxed{\text{No!}}$ Let's see what the contrary assumption actually is.

We are assuming

$\neg$ (for any integers m and n, $2 \nmid mn$ implies m is odd and n is odd.)

What does this mean?

For some integers m and n, $2 \nmid mn \wedge (2 \mid m \vee 2 \mid n)$

So a counter-example does NOT give us a contradiction.

As we said earlier, these are called quantifiers and we will discuss them in details, later.

**Pf of ($\Leftarrow$).** $\left.\begin{array}{l} m \text{ is odd} \\ n \text{ is odd} \end{array}\right\} \overset{?}{\Longrightarrow} mn$ is odd.

(Direct proof) $m$ is odd $\Rightarrow$ for some integer $k$,
$$m = 2k+1 \quad \text{I}$$
$n$ is odd $\Rightarrow$ for some integer $\ell$,
$$n = 2\ell+1 \quad \text{II}$$
$$\text{I and II} \Rightarrow mn = (2k+1)(2\ell+1)$$
$$= 4k\ell + 2k + 2\ell + 1$$
$$= 2\underbrace{(2k\ell + k + \ell)}_{\text{is integer}} + 1$$

So $mn = 2k' + 1$ for some integer $k'$,

which implies $mn$ is odd. ∎

In the previous lecture I mentioned some mathematician – Kronecker – said that "God made the integers, all else is the work of man."

The reason he says that integers are NOT man-made is

## The Induction Principle

Let's start with a few examples. We will cheat a bit at first.

**Ex.** Find $1 + 3 + 5 + \cdots + (2n-1)$.

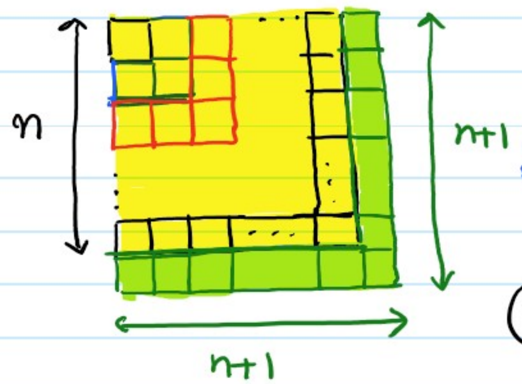
Trying to come-up with a conjecture (inductive reasoning)

$1 = 1$ , $1+3 = 4$ , $1+3+5 = 9$ ,

$(n=1)$ $(n=2)$ $(n=3)$

Any guess? Now you can form a conjecture:

$$1+3+5+\cdots+(2n-1) = n^2 .$$




New layer

building upon we have already built.

(This is almost a proof.)

Ex. What is $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ ?

The three dots says that a process should be repeated infinitely many times:

$\sqrt{2}$ , $\sqrt{2+\sqrt{2}}$ , $\sqrt{2+\sqrt{2+\sqrt{2}}}$ , ...

How can I give it to a computer to this for me?

$a_{n+1} = \sqrt{2+a_n}$ , $a_1 = \sqrt{2}$.

. Show $a_n < 2$ for any positive integer $n$.

Let's try small numbers.

$a_1 = \sqrt{2} < 2$

In class I did not show why $\sqrt{2} < 2$.

Here is a proof:

Here is a proof:

$1 < 2 \Rightarrow \sqrt{1} < \sqrt{2}$ (we have seen $x^2 \le y^2 \Leftrightarrow |x| \le |y|$.)

$\Rightarrow \sqrt{2} \cdot \sqrt{1} < \sqrt{2} \cdot \sqrt{2}$ ($\sqrt{2} > 0$)

$\Rightarrow \sqrt{2} < 2$.

How about $\sqrt{2+\sqrt{2}} < 2$ ? Backward arguement.

$$\sqrt{2+\sqrt{2}} < 2 \Leftarrow 2+\sqrt{2} < 4 \Leftarrow \sqrt{2} < 2$$

How about $\sqrt{2+\sqrt{2+\sqrt{2}}} < 2$ ?

$$\sqrt{2+\sqrt{2+\sqrt{2}}} < 2 \Leftarrow 2+\sqrt{2+\sqrt{2}} < 4 \Leftarrow \sqrt{2+\sqrt{2}} < 2$$

Can we always go one step forward?

$$a_{n+1} < 2 \Leftarrow \sqrt{2+a_n} < 2 \Leftarrow 2+a_n < 4 \Leftarrow a_n < 2.$$

Yes! So we have this "infinite chain of implications"

$a_1 < 2$

$a_1 < 2 \Rightarrow a_2 < 2 \Rightarrow a_3 < 2 \Rightarrow \cdots$

So it should be true.

In fact this is called the induction principle.

Here is the formal formulation of the induction principle:

$P(1)$ is true $\}$

$P(1)$ is true

For any positive integer $k$, $P(k) \Longrightarrow P(k+1)$

<u>implies</u>, for any integer $n$, $P(n)$ is true.

In the next lecture we will see the formal way of writing a proof-by-induction.