Math 109: The second exam.
Instructor: A. Salehi Golsefidy

Name: Solution ....

PID: ....

11/18/2015

1. Write your Name and PID on the front of your exam sheet.
2. No calculators or other electronic devices are allowed during this exam.
3. Show all of your work; no credit will be given for unsupported answers.
4. Read each question carefully to avoid spending your time on something that you are not supposed to (re)prove.
5. Ask me or a TA when you are unsure if you are allowed to use certain fact or not.
6. Good luck!

Question	Points	Score
1	10	
2	8	
3	10	
4	5	
5	7	
Total:	40	

1. Let $A = \{4k \mid k \in \mathbb{Z}\}$ and $B = \{n \in \mathbb{Z} \mid 6|n\}$. (For each part justify your answer. for the first two parts your justification can be rather brief.)
   (a) (2 points) Find the smallest positive element of $A \cap B$.
   (b) (3 points) Find the smallest positive element of $A \triangle B$.
   (c) (5 points) Let $f : A \times B \to \mathbb{Z}$, $f((m, n)) = m + n$. Is $1 \in \mathrm{Im}(f)$?

(a) We are looking for smallest positive integer that is a multiple of 4 and 6. So the answer is 12.

(b) We are looking for smallest positive integer that is either a multiple of 4 or a multiple of 6, but not a multiple of both 4 and 6. So the answer is 4.

[For (a) and (b), you could write

$A = \{\cdots, -4, 0, 4, 8, 12, \cdots\}$

$B = \{\cdots, -6, 0, 6, 12, 18, \cdots\}$

So as we can see 12 is the smallest positive integer in $A \cap B$, and $4 \in (A \cup B) \setminus (A \cap B) = A \triangle B$ is the smallest positive integer in $A \triangle B$.]

(c) $1 \in \mathrm{Im}(f) \iff \exists\, x, y \in \mathbb{Z},\ 1 = 4x + 6y$.

This cannot happen as 1 is odd and $2(2x+3y)$ is even. So $1 \notin \mathrm{Im}(f)$.

2. (8 points) Let A, B, and C be sets. Prove that $(A\cup B)\setminus(A\cup C) = B\setminus(A\cup C)$.

<u>Proof 1</u>.

$$\begin{aligned}
(A\cup B)\setminus(A\cup C) &= (A\cup B)\cap(A\cup C)^c \\
&= (A\cup B)\cap(A^c\cap C^c) \\
&= \big((A\cup B)\cap A^c\big)\cap C^c \\
&= \big((A\cap A^c)\cup(B\cap A^c)\big)\cap C^c \\
&= \big(\varnothing\cup(B\cap A^c)\big)\cap C^c \\
&= B\cap(A^c\cap C^c) \\
&= B\cap(A\cup C)^c \\
&= B\setminus(A\cup C).
\end{aligned}$$

<u>Proof 2</u>.

$$\begin{aligned}
x\in(A\cup B)\setminus(A\cup C) &\iff x\in A\cup B \wedge x\notin A\cup C \\
&\iff (x\in A \vee x\in B)\wedge(x\notin A\wedge x\notin C) \\
&\iff [(x\in A \vee x\in B)\wedge x\notin A]\wedge x\notin C \\
&\iff [\underbrace{(x\in A\wedge x\notin A)}_{\text{always true}}\vee(x\in B\wedge x\notin A)]\wedge x\notin C \\
&\iff x\in B \wedge x\notin A \wedge x\notin C \\
&\iff x\in B \wedge x\notin A\cup C \\
&\iff x\in B\setminus(A\cup C).
\end{aligned}$$

<u>Proof 3</u>.

$x\in A$	$x\in B$	$x\in C$	$x\in A\cup B$	$x\in A\cup C$	$x\in(A\cup B)\setminus(A\cup C)$ (because of II, III)	$x\in B\setminus(A\cup C)$ (because of I, III.)
T	T	T	T	T	F	F
T	T	F	T	T	F	F
T	F	T	T	T	F	F
T	F	F	T	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	F	T	F	F
F	F	F	F	F	F	F
	I		II	III		

same columns.

3. For a real sequence $a_1, a_2, \ldots$, we say $\lim_{n\to\infty} a_n = L$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+, n \geq N \Rightarrow |a_n - L| < \varepsilon.$$

(a) (6 points) Use quantifiers to say what it means to say $\lim_{n\to\infty} a_n$ does not exist.

(b) (4 points) Prove that $\lim_{n\to\infty}(-1)^n$ does not exist. (Hint: use proof by contradiction and assume $\lim_{n\to\infty}(-1)^n = L$ for some $L \in \mathbb{R}$.)

(a) $\forall L \in \mathbb{R}, \exists \varepsilon > 0, \forall N \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+, n \geq N \wedge |a_n - L| \geq \varepsilon$.

(b) Suppose to the contrary that $\lim\limits_{n\to\infty} (-1)^n = L$.

So $\forall \varepsilon > 0, \exists N \in \mathbb{Z}^+, n \geq N \Rightarrow |(-1)^n - L| < \varepsilon$.

In particular, $\exists N \in \mathbb{Z}^+, n \geq N \Rightarrow |(-1)^n - L| < 1/2$.

$\Rightarrow |(-1)^N - L| < 1/2$ and $|(-1)^{N+1} - L| < 1/2$ (I)

Notice that $(-1)^N = 1 \Rightarrow (-1)^{N+1} = -1$

and $(-1)^N = -1 \Rightarrow (-1)^{N+1} = 1$. So (I) implies

$|1 - L| < 1/2$ and $|-1 - L| < 1/2$. Hence

$L - 1 > -1/2$ and $L + 1 < 1/2$. Therefore

$L > 1/2$ and $L < -1/2$ which is a contradiction.

4. (5 points) Let $f : \mathbb{R} \to \mathbb{R}^{\geq 0}, f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = x + 1$. Find the following functions if there are defined:

$$f \circ f, \quad f \circ g, \quad g \circ g, \quad g \circ f.$$

Justify your answers.

$\mathbb{R} \xrightarrow{f} \mathbb{R}^{\geq 0} \qquad \mathbb{R} \xrightarrow{g} \mathbb{R}$

- Since (codomain of f) $\neq$ (domain of g), $g \circ f$ is NOT defined.
- Since (codomain of f) $\neq$ (domain of f), $f \circ f$ is NOT defined,

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R}^{\geq 0} \end{tikzcd}$$

$$(f \circ g)(x) = f(g(x)) = f(x+1) = (x+1)^2$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "g \circ g"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$$(g \circ g)(x) = g(g(x)) = g(x+1) = (x+1)+1 = x+2.$$

5. (7 points) Let $f : X \to Y$ be a function. Suppose $g \circ f = I_X$, for some function $g : Y \to X$, where I_X is the identity function on X. Prove that f is injective.

$$
\begin{aligned}
f(x_1) = f(x_2) &\Rightarrow g(f(x_1)) = g(f(x_2)) \\
&\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2) \\
&\Rightarrow I_X(x_1) = I_X(x_2) \\
&\Rightarrow x_1 = x_2 .
\end{aligned}
$$