Tuesday, May 17, 2016 12:02 PM

1. Let $\psi(x) = \sum_{p \le x} \ln p$.

 (a) Prove that $\psi(x) = O(x)$.

 (b) Prove that $\psi(x) \sim x \iff$ Prime Number Theorem, i.e.

$$\pi(x) \sim x/\ln x.$$

[Hint: (a) We have already proved a stronger result.

(b) We have proved the following:

- $\{\lambda_n\}$ strictly increasing seq of positive integers
- $\{c_n\}$ a sequence of complex numbers
- $f: \mathbb{R}^+ \to \mathbb{C}$ differentiable

Let $C(t) := \sum_{\lambda_n \le t} c_n$. Then

$$\sum_{\lambda_n \le x} c_n f(\lambda_n) = C(x) f(x) - \int_{\lambda_1}^{x} C(t) f'(t)\, dt.$$

- Let $\lambda_n := p_n$ be the n^{th} prime number, $f(x) = \frac{1}{\ln x}$, and $c_n = \ln p_n$.
- You have proved it before that $\int_2^x \frac{1}{(\ln t)^2}\, dt \ll \frac{x}{(\ln x)^2}$.]

2. Prove that there is a holomorphic function h on an open neighborhood U of $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \ge 1\}$ such that

$$h(s) + \frac{1}{s-1} = \sum_p \frac{\ln p}{p^s} \quad \text{for } \mathrm{Re}(s) > 1.$$

[Hint. In the previous HW, you proved that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1$$

$$= \sum_p \frac{\ln p}{p^s} + \sum_{p,\, m \ge 2} \frac{\ln(p)}{p^{ms}} \qquad \text{for } \mathrm{Re}(s) > 1.$$

$p \quad p^{\smile} \qquad p, m\geq 2 \qquad p^{m\smile}$

① Show that $g_1(s) := \sum_{p, m\geq 2} \frac{\ln(p)}{p^{ms}}$ is holomorphic on an open neighborhood of $\{z\in\mathbb{C} \mid Re(z)\geq 1\}$.

② $\sum_p \frac{\ln p}{p^s}$ is holomorphic on $Re(s)>1$.

③ $\exists$ a holomorphic function g_2 on an open neighborhood U of $\{z\in\mathbb{C} \mid Re(z)\geq 1\}$ such that $\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + g_2(s)$

[This is the important step. You should use:

ⓐ $\zeta(s) = \frac{1}{s-1} + \phi(s)$ where ϕ is holomorphic on $Re(s)>0$.

ⓑ $\zeta(s)\neq 0$ if $Re(s)\geq 1$ (Previous HW.)]

④ Let $h(s) := g_2(s) - g_1(s)$ on U, and conclude

$$\sum_p \frac{\ln p}{p^s} = \frac{1}{s-1} + h(s) \quad \text{for } Re(s)>1.\ ]$$

3 ⓐ Prove that $\sum_p \frac{\ln p}{p^s} = s\int_1^\infty \frac{\psi(t)}{t^{s+1}}\,dt$ for $Re(s)>1$.

[Hint. Use the mentioned "integration by-part" in the hint of problem 1. And $\psi(t)=0$ if $1\leq t<2$.

ⓑ Conclude $\int_1^\infty \frac{\psi(t)-t}{t^{s+1}}\,dt = \frac{h(s)-1}{s}$ for $Re(s)>1$,

and so the LHS has a holomorphic extension on U, a neighborhood of $\{z\in\mathbb{C} \mid Re(z)\geq 1\}$.

§ Suppose as in Landau theorem we manage to use the holomorphic extension of $\int_1^\infty \frac{\psi(t)-t}{t^{s+1}}dt$ to conclude this is a convergent integral even at $\underline{\underline{s=1}}$. I.e.

$$\int_1^\infty \frac{\psi(t)-t}{t^2}\,dt < \infty.$$

4. Prove $\psi(x)\sim x$.

[Hint. We have to show

$$\forall \varepsilon>0, \quad x \underset{\varepsilon}{\gg} 1 \Rightarrow \quad (1-\varepsilon)\, x \le \psi(x) \le (1+\varepsilon)\, x$$

If not, then either there is a sequence

$$x_1 < x_2 < \cdots, \quad x_n \to \infty \quad \text{and} \quad \psi(x_n) > (1+\varepsilon)\, x_n$$

or there is a sequence $y_1 < y_2 < \cdots$, $y_n \to \infty$ and

$$\psi(y_n) < (1-\varepsilon)\, y_n .$$

Now we would like to get a contradiction:

$$\int_{x_n}^{(1+\varepsilon)x_n} \frac{\psi(t)-t}{t^2}\, dt \underset{(?)}{\ge} \int_{x_n}^{(1+\varepsilon)x_n} \frac{(1+\varepsilon)x_n - t}{t^2}\, dt$$

$$\underset{?}{=} \varepsilon - \ln(1+\varepsilon) \qquad \circledast$$

Since $\int_1^\infty \frac{\psi(t)-t}{t^2}\, dt$ is convergent,

$$\lim_{n\to\infty} \int_{x_n}^{(1+\varepsilon)x_n} \frac{\psi(t)-t}{t^2}\, dt = ? \quad , \text{ which contradicts } \circledast \ (?)$$

The other part is similar.]

So modulo § you have proved the Prime Number Theorem. Statements similar to § is called Tauberian Theorems.

Here we used Newman's simple proof of prime number theorem. I have used the following article by Zagier:

Zagier, Newman's Short Proof of Prime Number Theorem,

The American Mathematical Monthly, 104, no. 8, (Oct., 1997) 705–708.