1. Prove that $-\dfrac{\zeta'(s)}{\zeta(s)} = \displaystyle\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$ if $\mathrm{Re}(s) > 1$,

where
$$\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^m \\ 0 & \text{otherwise}. \end{cases}$$

[Hint. Use Euler product formula and compute $\dfrac{d}{ds}(\ln \zeta(s))$.

Justify your equations.]

2. Show that $\displaystyle\sum_{d \le x} \mu(d) \lfloor x/d \rfloor = 1$ for $x \in \mathbb{R}^{\ge 1}$.

Deduce $\left| \displaystyle\sum_{d \le x} \mu(d)/d \right| \le 1$ for $x \in \mathbb{R}^{\ge 1}$.

3. Prove that $n! = m^k$ is impossible in integers $k > 1$, $m > 1$, $n > 1$.

(Hint. You have already solved this!)

4. In this exercise you see the beautiful treatment of Merten to show that ζ-function has no zero on the line $\mathrm{Re}(s) = 1$. This result is at the heart of proof of Prime Number Theorem.

(a) $\zeta(s) \ne 0$ if $\mathrm{Re}(s) > 1$.

[Hint. Use $\zeta(s) = e^{-\sum_p \ln(1 - 1/p^s)}$.]

(b) Using the result that you proved before, show that
$$\zeta(s) = \frac{1}{1-s} + \phi(s)$$
where ϕ is holomorphic on $\mathrm{Re}(s) > 0$.

(c) Show that $|\zeta(\sigma)| \ll \dfrac{1}{\sigma - 1}$ for $0 < \sigma - 1 \ll 1$.

(d) Suppose $\zeta(1 + it_0) = 0$. Prove that, for $0 < \sigma - 1 \ll 1$,

$|\qquad|$

$|\zeta(\sigma+i\gamma)| \ll \sigma-1$.

(e) Deduce that (Same t_0 as (d))

$$|\zeta(\sigma+it_0)^4\zeta(\sigma)^3\zeta(\sigma+2it_0)| \xrightarrow[\sigma\to1^+]{} 0 .$$

(f) Show that, for $\mathrm{Re}(s)>1$,

$$\ln|\zeta(s)| = \sum_{n=1}^{\infty} \lambda(n)\,\mathrm{Re}(n^{-s}) .$$

where $\lambda(n) = \begin{cases} \frac{1}{m} & \text{if } n=p^m \\ 0 & \text{otherwise.} \end{cases}$

[Hint. $\ln|\zeta(s)| = \mathrm{Re}(\ln\zeta(s)) = \mathrm{Re}\left(-\sum_p \mathrm{Ln}(1-1/p^s)\right)$

$$= \sum_{p,m} \mathrm{Re}\left(\frac{1}{mp^{ms}}\right) .]$$

(g) Prove $\ln|\zeta(\sigma+it_0)^4\zeta(\sigma)^3\zeta(\sigma+2it_0)| \geq 0$

[Hint. By part (f) (Same t_0 as (d))

$$\ln|\zeta(\sigma+it_0)^4\zeta(\sigma)^3\zeta(\sigma+2it_0)|$$

$$=\sum_n \lambda(n)\left(4\,\mathrm{Re}(n^{-\sigma-it_0}) + 3\,\mathrm{Re}(n^{-\sigma}) + \mathrm{Re}(n^{-\sigma-2it_0})\right)$$

$$=\sum_n \lambda(n)\, n^{-\sigma}\left(4\cos(t_0\ln n) + 3 + \cos(2t_0\ln n)\right)$$

(Let $\theta_n := t_0\cdot\ln n$) $\rightsquigarrow$ $= \sum_n \lambda(n)\, n^{-\sigma}\left(\cos(2\theta_n) + 4\cos(\theta_n) + 3\right)$.

<u>And</u>

$$\cos(2\theta)+4\cos\theta+3 = 2\cos^2\theta - 1 + 4\cos\theta + 3$$

$$= 2(\cos^2\theta + 2\cos\theta + 1)$$

$$= 2(\cos\theta+1)^2 \geq 0 \quad .]$$

(h) Conclude that $\zeta(s)$ does not have a zero on

$$\{s\in\mathbb{C} \mid \mathrm{Re}(s)\geq 1\} .$$