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1. (a) Prove that $\varphi(n) \gg \dfrac{n}{\ln(\ln n)}$.

(b) Prove that there is a sequence $n_1 < n_2 < \cdots$ of positive integers such that

$$\varphi(n_i) \ll \frac{n_i}{\ln(\ln n_i)}.$$

[φ is the Euler-phi function.]

[Hint (a) $\dfrac{n}{\varphi(n)} = \displaystyle\prod_{p \mid n} \frac{1}{1-1/p}$.

$$\Longrightarrow \ln\left(\frac{n}{\varphi(n)}\right) = \sum_{p\mid n} -\ln\left(1-\frac{1}{p}\right)$$

$$= \sum_{p\mid n} \sum_{m=1}^{\infty} \frac{1}{m p^m}$$

$$= \sum_{p\mid n} \frac{1}{p} + O(1).$$

For any level D we have

$$\sum_{p\mid n} \frac{1}{p} \le \sum_{p\le D} \frac{1}{p} + \sum_{\substack{p\mid n \\ D\le p}} \frac{1}{p}$$

$$\le \ln \ln D + \frac{\log_D n}{D} + O(1).$$

$$= \ln \ln D + \frac{\ln n}{(\ln D)(D)} + O(1)$$

Let $D = \ln n$.

(b) Let $p_1 < p_2 < \cdots$ be the sequence of prime numbers.

$$\ln \frac{p_1 \cdots p_r}{\varphi(p_1 \cdots p_r)} = \sum_{p\le p_r} -\ln\left(1-\frac{1}{p_i}\right)$$

$$= \sum_{p\le p_r} \frac{1}{p} + O(1)$$

$$\ge \ln \ln p_r - O(1) \ge \ln \ln \ln n_r - O(1)$$

$$\boxed{n_r := p_1 \cdots p_r < 4^{p_r} \implies p_r > \log_4 n_r} \quad . \;]$$

2. Let $\theta(n)$ be the number of positive integers $m \le n$ such that

$$(m,n) = (m+1, n) = 1.$$

(a) Prove that $\quad \theta(n) = n \prod_{p \mid n} \left(1 - \frac{2}{p}\right).$

(b) Prove that $\quad \theta(n) \gg n (\ln \ln n)^{-2}$ if $2 \nmid n$.

[Notice that $\theta(n) = 0$ if $2 \mid n$.]

[Hint (a) For any $p \mid n$, let $A_p := \{ m \in \mathbb{Z}^+ \mid m \le n, \ m \overset{p}{\equiv} 0 \text{ or } -1 \}$.

$\Rightarrow \theta(n) = \left| \{1, \dots, n\} \setminus \bigcup_{p \mid n} A_p \right|$. Use inclusion-exclusion.

$m \in A_{p_1} \cap A_{p_2} \cap \cdots \cap A_{p_k} \iff m \overset{p_i}{\equiv} 0$ or -1 and $m \le n$

By Chinese remainder theorem, m has 2^k possibilities modulo $p_1 p_2 \cdots p_k$

$\Rightarrow |A_{p_1} \cap \cdots \cap A_{p_k}| = 2^k \cdot n / p_1 p_2 \cdots p_k$.

There are two ways to finish part (a). A cleaner way to write it:

Hence $\theta(n) = \sum_{q \mid n} \mu(q) 2^{\omega(q)} \frac{n}{q}$ where μ is the Möbius function and $\omega(q)$ is the number of distinct prime factors of q.

Hence θ is multiplicative as it is the convolution of

$$n \mapsto \mu(n) 2^{\omega(n)} \quad \text{and} \quad n \mapsto n.$$

And $\theta(p^m) = p^m - 2p^{m-1} = p^m\left(1 - \frac{2}{p}\right)$. Then finish the proof.

(b) $\ln \frac{n}{\theta(n)} = \prod_{p \mid n} - \ln\left(1 - \frac{2}{p}\right)$ and continue as in problem 1 (a).]

3. Let n be an odd square-free positive integer and $\zeta_n = e^{\frac{2\pi i}{n}}$.

Prove that $\quad \log \prod^{n} \prod^{n} \left(1 + \zeta^{ab}\right) - \prod (2p-1).$

Prove that $\log_2 \prod_{a=1}^{n} \prod_{b=1}^{n} (1+\zeta_n^{ab}) = \prod_{p\mid n} (2p-1)$.

[Hint: Let $M_k := \{ \zeta_k^m \mid 1 \le m \le k \}$.

Step 1. Notice that $z \longmapsto z^a$ induces an $\gcd(a,n)$-to-1 map from M_n to $M_{n/\gcd(a,n)}$.

Step 2. Suppose $\gcd(a,n)=d$. Using Step 1 show that

$$\prod_{\zeta\in M_n} (x-\zeta^a) = (x^{n/d}-1)^d .$$

Step 3. Let $B_d = \{ a\in\mathbb{Z}^+ \mid 1\le a\le n,\ \gcd(a,n)=d \}$.

$\Rightarrow \prod_{a=1}^{n} * = \prod_{d\mid n} \prod_{a\in B_d} *$, and $|B_d| = \varphi(n/d)$.

Step 4. By steps 2 and 3 deduce

$$\prod_{a=1}^{n} \prod_{\zeta\in M_n} (x-\zeta^a) = \prod_{d\mid n} (x^{n/d}-1)^{d\varphi(n/d)} .$$

Step 5.

$$\begin{aligned}\prod_{a=1}^{n} \prod_{b=1}^{n} (1+\zeta_n^{ab}) &= (-1)^{n^2} \prod_{a=1}^{n} \prod_{\zeta\in M_n} (-1-\zeta^a) \\ &= (-1)^{n^2} \prod_{d\mid n} ((-1)^{n/d}-1)^{d\varphi(n/d)} \\ &\overset{?}{=} (-1)^{n^2} \prod_{d\mid n} (-2)^{d\varphi(n/d)} \\ &= (-1)^{n^2+\sum_{d\mid n} d\varphi(n/d)}\ 2^{\sum_{d\mid n} d\varphi(n/d)}\end{aligned}$$

Step 6. $\sum_{d\mid n} d\,\varphi(n/d) = (\mathrm{id}\cdot * \varphi)(n)$ is a multiplicative function and $(\mathrm{id}\cdot *\varphi)(p) = \varphi(p)+p = 2p-1$.

So for a square-free number n

$\sum$

$$\sum_{d|n} d\,\varphi(n/d) = \prod_{p|n} (2p-1)$$

in particular it is odd.

<u>Step 7</u>. Finish the proof.]