Wednesday, April 20, 2016 10:58 PM

1. Show that, if $n, k \in \mathbb{Z}$, $n > 1$, $k > 0$, then the number

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k}$$

is not an integer.

[Hint. If $k < n$, then $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k} < \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{k+1 \text{ many}}$

$$= \frac{k+1}{n} \leq 1.$$

. If $k \geq n$, use Bertrand hypothesis to find a prime p

s.t. $n \leq \left\lfloor \frac{n+k}{2} \right\rfloor < p \leq n+k$. Take the common

denominator to conclude that p divides the denomi., but

not the numerator.]

2. Prove that $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{k(k-1)}$ is convergent.

3. Show that, for any $n \in \mathbb{Z}^{>1}$, there is a prime p such that

$$v_p(n!) = 1.$$

4. There are n lights and n students. Both lights and students are labeled from 1 to n. Students go from 1 to n, and the i^{th} student changes the status of the lights that are multiple of i.

Ex. $n = 4$: 0 0 0 0 $\xrightarrow{1^{st}}$ 1 1 1 1

$\xrightarrow{2^{nd}}$ 1 0 1 0

$\xrightarrow{3^{rd}}$ 1 0 0 0

$\xrightarrow{4^{th}}$ 1 0 0 1

Which lights will be on at the end?

5. (a) $\int^{\infty} \frac{dt}{s} = \frac{1}{1}$ for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

5. (a) $\displaystyle\int_1^\infty \frac{dt}{t^s} = \frac{1}{s-1}$ for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

[Hint. $|t^{1-s}| = t^{1-\mathrm{Re}(s)} \xrightarrow[t\to\infty]{} 0$.]

(b) For any $n \in \mathbb{Z}^{\geq 1}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$,

let $\displaystyle g_n(s) = \int_n^{n+1} \frac{1}{n^s} - \frac{1}{t^s}\, dt.$

Prove that $\displaystyle\sum_{n=1}^{\infty} g_n(s)$ is absolutely convergent.

[Hint. $\displaystyle\left|\frac{1}{n^s} - \frac{1}{t^s}\right| = \left|\int_n^t \frac{s}{u^{s+1}}\, du\right| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}} |t-n|$

$$\leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$$

$\displaystyle\Rightarrow |g_n(s)| \leq \frac{|s|}{n^{\mathrm{Re}(s)+1}}.$]

(c) Let $\displaystyle g(s) = \sum_{n=1}^{\infty} g_n(s)$. Prove that for $\mathrm{Re}(s) > 1$

we have $\displaystyle\zeta(s) = \frac{1}{s-1} + g(s).$

(c) Let $g(s) = \sum_{n=1}^{\infty} g_n(s)$ for $\mathrm{Re}(s) > 0$. Prove that for $\mathrm{Re}(s) > 1$, we have

$$\zeta(s) = \frac{1}{s-1} + g(s).$$