Friday, April 15, 2016 12:06 AM

1. (a) Prove that, for $x, y \in \mathbb{R}$, we have

$$\lfloor 2x \rfloor + \lfloor 2y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor + \lfloor x+y \rfloor$$

(b) Use part (a) to prove that

$$\forall m, n \in \mathbb{Z}^+, \quad \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!} \in \mathbb{Z}.$$

(Hint (a) $x = \lfloor x \rfloor + \alpha$ and $y = \lfloor y \rfloor + \beta$ where $0 \leq \alpha, \beta < 1$.

$\Rightarrow \lfloor 2x \rfloor = 2\lfloor x \rfloor + \lfloor 2\alpha \rfloor$, $\lfloor 2y \rfloor = 2\lfloor y \rfloor + \lfloor 2\beta \rfloor$,

$\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor + \lfloor \alpha + \beta \rfloor$.

So it is enough to show $\lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor \geq \lfloor \alpha + \beta \rfloor$.

The main advantage is that $0 \leq \alpha, \beta < 1$.

Now notice that $\alpha + \beta \geq 1 \Rightarrow$ either $2\alpha \geq 1$ or $2\beta \geq 1$.

(b) Use $v_p(m!) = \sum_{i=1}^{\infty} \lfloor \frac{m}{p^i} \rfloor$, and the fact that

$m \mid n \iff \forall p \in \mathcal{P},\ v_p(m) \leq v_p(n)$.)

2. Prove that $\lfloor nx \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{n} \rfloor + \cdots + \lfloor x + \frac{n-1}{n} \rfloor$.

(Hint. By division algorithm, $\exists\, q, r \in \mathbb{Z}$,

- $\lfloor nx \rfloor = nq + r$
- $0 \leq r < n$

$\Rightarrow \quad nq + r \leq nx < nq + r + 1$

$\Rightarrow \quad q + \frac{r}{n} \leq x < q + \frac{r+1}{n}$

$\boxed{\text{Case 1}}$ $n - r \leq i \leq n-1$ (there are r many such i's.)

$$q+1 \leq q + \frac{r+i}{n} \leq x + \frac{i}{n} < q + \frac{r+i+1}{n} < q+2$$

$\Rightarrow \lfloor x + \frac{i}{n} \rfloor = q + \underline{\underline{1}}.$

$\boxed{\text{Case 2}}$ $0 \leq i \leq n-r-1$

$\boxed{\text{Case 2}}$ $\quad 0 \le i \le n-r-1$

$$q \le q + \frac{r+i}{n} \le x + \frac{i}{n} < q + \frac{r+i+1}{n} \le q+1$$

$$\Rightarrow \lfloor x + \tfrac{i}{n} \rfloor \cdot \quad)$$

3. (Beaty sequences) Suppose $\alpha, \beta > 1$ are two irrational numbers. $\{\lfloor n\alpha \rfloor \mid n \in \mathbb{Z}^+\}$ and $\{\lfloor n\beta \rfloor \mid n \in \mathbb{Z}^+\}$ form a partition of $\mathbb{Z}^+$

if and only if $\quad \dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1.$

(Hint. ($\Rightarrow$) Since $\alpha, \beta > 1$, $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$ are strictly increasing sequences. Since $\{\lfloor n\alpha \rfloor \mid n \in \mathbb{Z}^+\} \sqcup \{\lfloor n\beta \rfloor \mid n \in \mathbb{Z}^+\} = \mathbb{Z}^+$, for any positive integer m we have

$$\left|\{\lfloor n\alpha \rfloor \mid n \in \mathbb{Z}^+,\ \lfloor n\alpha \rfloor \le m\}\right| + \left|\{\lfloor n\beta \rfloor \mid n \in \mathbb{Z}^+,\ \lfloor n\beta \rfloor \le m\}\right| = m.$$

Find out and , divide by m and let m go to infinity. [essentially we are finding density of sets, and they should add up to 1.]

($\Leftarrow$) List all the numbers of the form $\frac{n}{\alpha}$ and $\frac{m}{\beta}$.

Step 1. They are distinct.

$$\frac{n}{\alpha} = \frac{m}{\beta} \Rightarrow \frac{n+m}{\alpha} = m \Rightarrow \alpha \in \mathbb{Q}, \text{ which is a contradiction.}$$

Step 2. Find out how many points in this list are $\le \frac{n}{\alpha}$.

- $\frac{1}{\alpha}, \frac{2}{\alpha}, \dots, \frac{n}{\alpha} \mapsto n.$
- $\frac{m}{\beta} \le \frac{n}{\alpha} \Leftrightarrow \frac{m}{\beta} + \frac{n}{\beta} \le \frac{n}{\alpha} + \frac{n}{\beta} = n$

$$\Leftrightarrow m \le n\beta - n.$$

$$\Rightarrow \quad n + \lfloor n\beta - n \rfloor = \lfloor n\beta \rfloor \qquad (\beta \text{ is irrational}).$$

Similarly $\frac{m}{\beta}$ is the $\lfloor m\alpha \rfloor^{th}$ number in this list.

h the proof.

(Any number

Step 3. Finish the proof.

(Any number in this list is either $\lfloor n\alpha \rfloor^{th}$ number in the list or the $\lfloor m\beta \rfloor^{th}$ number of the list, and only one of them.) .)

4. Prove that $li(x) = \int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} + O\left(\frac{x}{(\ln x)^2}\right)$.

(Hint. $$\int_2^x \frac{dt}{\ln t} = \frac{t}{\ln t}\Big|_2^x + \int_2^x \frac{t}{t(\ln t)^2}\,dt = \frac{x}{\ln x} + \int_2^x \frac{dt}{(\ln t)^2} - \frac{2}{\ln 2}$$

$$\begin{cases} u = (\ln t)^{-1} \Rightarrow du = -(\ln t)^{-2}\frac{dt}{t}. \\ dv = dt \Rightarrow v = t \end{cases}$$

Use integration by-part $\int u\,dv = uv - \int v\,du$

$$\int_2^x \frac{dt}{(\ln t)^2} = \int_2^{\sqrt{x}} \frac{dt}{(\ln t)^2} + \int_{\sqrt{x}}^x \frac{dt}{(\ln t)^2} \ll \sqrt{x} + \frac{x}{(\ln x)^2}$$

$$\ll \frac{x}{(\ln x)^2}\,.)$$

(Remark. Using a similar argument one can get

$$\int_2^x \frac{dt}{\ln t} = \frac{x}{\ln x} + 1!\,\frac{x}{(\ln x)^2} + \cdots + q!\,\frac{x}{(\ln x)^q} + O\left(\frac{x}{(\ln x)^{q+1}}\right).\,)$$

($li(x)$ is called logarithmic integral.)