Tuesday, April 5, 2016 10:37 AM

1. Prove that for any prime p and positive integers m and n

$$v_p(\gcd(m,n)) = \min\{v_p(m), v_p(n)\}$$

$$v_p(\operatorname{lcm}(m,n)) = \max\{v_p(m), v_p(n)\}$$

Conclude that $\gcd(m,n)\cdot \operatorname{lcm}(m,n) = m\cdot n$.

2. Let $\Lambda(n) = \begin{cases} \log p & \text{if } n \text{ is a power of any prime } p \\ 0 & \text{otherwise.} \end{cases}$

(i) Show that $\log n = \sum_{d|n} \Lambda(d)$.

(ii) Deduce that $\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$.

3. Let $\omega(n)$ be the number of distinct primes dividing n.

(i) Prove that $2^{\omega(n)} \le \tau(n) \le n$.

(ii) Prove that $\phi(n) \ge n \prod_{k=2}^{\omega(n)+1}\left(1-\frac{1}{k}\right) = \frac{n}{\omega(n)+1}$.

(iii) Conclude $\phi(n) > \frac{cn}{\log n}$ for some suitable constant $c>0$ and any $n \in \mathbb{Z}^{\ge 2}$.

4. Suppose $f, g: \mathbb{Z}^+ \to \mathbb{C}$ are two functions.

Suppose there is a positive integer k such that for any $n \in \mathbb{Z}^+$

$$|f(n)|, |g(n)| \le n^k$$

(we say that f and g have at most polynomial growth.)

(i) Prove that, if $\mathrm{Re}(s) > k+1$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

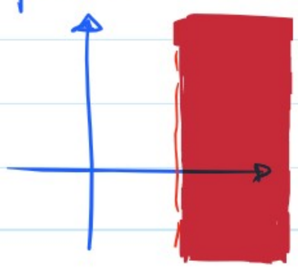
are absolutely convergent.

(ii) Prove that, for $\mathrm{Re}(s) > k+1$, we have

$$\left(\sum^{\infty} \frac{f(n)}{}\right)\left(\sum^{\infty} \frac{g(n)}{}\right) = \sum^{\infty} \frac{(f*g)(n)}{}.$$

$$\left(\sum_{n=1} \frac{f(n)}{n^s}\right)\left(\sum_{n=1} \frac{g(n)}{n^s}\right) = \sum_{n=1} \frac{(f*g)(n)}{n^s}.$$

[$\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ is called the zeta-function associated with f, and it is denoted by $\zeta_f(s)$. So the above equality implies that, if f and g do not grow faster than a polynomial, in a half-plane



we have $\zeta_f(s)\,\zeta_g(s) = \zeta_{f*g}(s)$.

Notice $\zeta_I(s) = 1$; $\zeta_{\mathbb{1}}(s) = \underbrace{\zeta(s)}_{\text{the usual zeta-function.}}$

So we have
- $\zeta_\tau(s) = \zeta_{\mathbb{1}*\mathbb{1}}(s) = \zeta(s)^2$
- $\zeta_\mu(s) = \zeta(s)^{-1}$.]

5. Suppose $f : \mathbb{Z}^+ \to \mathbb{C}$ is a multiplicative function, and there is $k \in \mathbb{Z}^+$ such that $|f(n)| \le n^k$ for any $n \in \mathbb{Z}^+$.

Prove that

$$\zeta_f(s) = \prod_{p \in \mathcal{P}} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right)$$

if $\mathrm{Re}(s) > k+1$. In particular, if f is strongly multiplicative, i.e. $f(mn) = f(m)f(n)$ for any $m, n \in \mathbb{Z}^+$, then

$$\zeta_f(s) = \prod_{p \in \mathcal{P}} \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

if $\mathrm{Re}(s) > k+1$.

[This is called Euler Product.]

[Two important properties of absolutely convergent series are the following:

① If $\displaystyle\sum_{n=1}^{\infty} |a_n| < \infty$, then for any bijection

$$f : \mathbb{Z}^+ \to \mathbb{Z}^+$$

∞

$\sum_{n=1} a_{f(n)}$ is convergent and $\sum_{n=1} a_{f(n)} = \sum_{n=1} a_n$.

② Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$. Suppose $\{I_1, I_2, \cdots\}$ is a partition of $\mathbb{Z}^+$. Then $\sum_{i=1}^{\infty} \left(\sum_{j \in I_i} a_j \right)$ is convergent and

$$\sum_{i=1}^{\infty} \left(\sum_{j \in I_i} a_j \right) = \sum_{i=1}^{\infty} a_i \text{ .]}$$