Tuesday, May 3, 2016 8:34 AM

1. (i) Say why $n \mapsto \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}$ is a multiplicative function.
(6 pts)

(ii) Deduce that $\frac{n}{\varphi(n)} = \sum_{d \mid n} \frac{\mu(d)^2}{\varphi(d)}$.
(6 pts)

2. Suppose p is prime, $a, m \in \mathbb{Z}^+$, and $p \nmid a$.

(i) If $p \nmid m$ and $v_p(a^t - 1) \geq 1$ for some $t \in \mathbb{Z}^+$, then
(5 pts)

$$v_p(a^{tm} - 1) = v_p(a^t - 1).$$

(ii) If $v_p(a^t - 1) \geq 2$, then $v_p(a^{tp} - 1) = v_p(a^t - 1) + 1$.
(3 pts)

(iii) If $v_p(a^{\operatorname{ord}_p a} - 1) \geq 2$, then, for any $n \in \mathbb{Z}^+$,
(6 pts)

either $v_p(a^n - 1) = 0$

or $v_p(a^n - 1) = v_p(a^{\operatorname{ord}_p a} - 1) + v_p(n)$.

3. (i) Prove that $v_p\binom{2n}{n} \leq \log_p n$ for any prime p.
(8 pts)

(ii) Prove that $v_p\binom{2n}{n} = 1$ for any prime $n < p \leq 2n$.
(4 pts)

(iii) Prove that
(8 pts)

$$n^{\pi(2n) - \pi(n)} \leq \binom{2n}{n} \leq (2n)^{\pi(2n)}$$

[This was Chebyshev's idea to show

$$\frac{n}{\ln n} \ll \pi(n) \ll \frac{n}{\ln n}.$$]

4. Write the outline of a proof of Bertrand's hypothesis.
(24 pts)